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Letter to the Editor

The Systemic Behavior of the Binding Energy of Heavy Isotopes

The ratio of the number of neutrons to the number of protons for stable heavy isotopes with $Z \geq 65$ is presented in Fig. 1. The ratio of the number of neutrons to the protons obtained for stable isotopes with high Z is ~1.84. This value results from the addition of two neutrons to each additional proton, a well-known effect due to the short range of nuclear attraction compared to long-range Coulomb repulsion. This phenomenon led us to consider the isotopes that have a $2Z - N$ = constant value, namely, isotopes with two additional neutrons to each additional proton. These isotopes are presented in Fig. 2. We can see from Fig. 2 that all heavy stable isotopes show this constant $2Z - N$ value behavior.

We have studied the $B.E./A$ for actinide isotopes ($Z \geq 88$) with the same $2Z - N$ values. Only experimental[1] values of the $B.E./A$ were considered. We then grouped the heavy isotopes into four groups: even(Z)-even(N), odd(Z)-odd(N), even(Z)-odd(N), and odd(Z)-even(N). Regarding the pairing effect, there should not be a difference in the $B.E./A$ between even(Z)-odd(N) and odd(Z)-even(N); however, we have considered them as two separate groups. The justification for separating even-odd and odd-even isotopes will be discussed later. The experimental $B.E./A$ for the isotopes with a constant $2Z - N$ value for each of the groups was plotted against their atomic number (Z) values. It was found that there is a very good linear relation between the $B.E./A$ and Z as demonstrated in Fig. 3 for even-even isotopes with $2Z - N = 38$. It was found that

$$B.E./A = -aZ + b \ , \tag{1}$$

where $a = 22.9$ and $b = 9676.9$ and with a correlation factor $R^2 = 0.99995$.

We have found similar relations between all the heavy isotopes. The other results for even-even isotopes with different $2Z - N$ values are given in Table I.

For the even(Z)-odd(N) isotopes with $2Z - N = 45$, the $B.E./A$ with respect to Z is given in Fig. 4. For the other $2Z - N$ values in this group, the results are presented in Table II. For the odd(Z)-even(N) and $2Z - N = 42$, the relation between $B.E./A$ and Z is given in Fig. 5, and for the other $2Z - N$ values, the results are presented in Table III. For the odd(Z)-odd(N) isotopes with $2Z - N = 45$, the results are presented in Fig. 6 and Table IV.

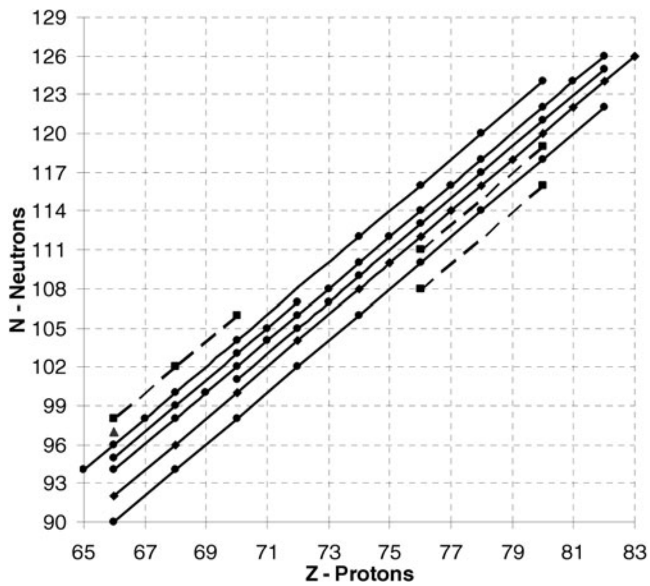

Fig. 2. The relations between the number of neutrons and protons for heavy ($Z \geq 65$) stable isotopes with $2Z - N$ = constant.

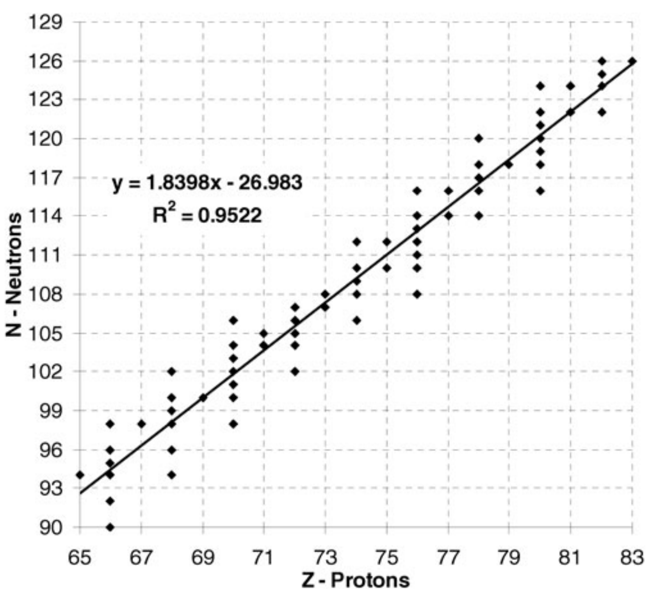

Fig. 1. The number of neutrons with respect to the number of protons for heavy ($Z \geq 65$) stable isotopes.

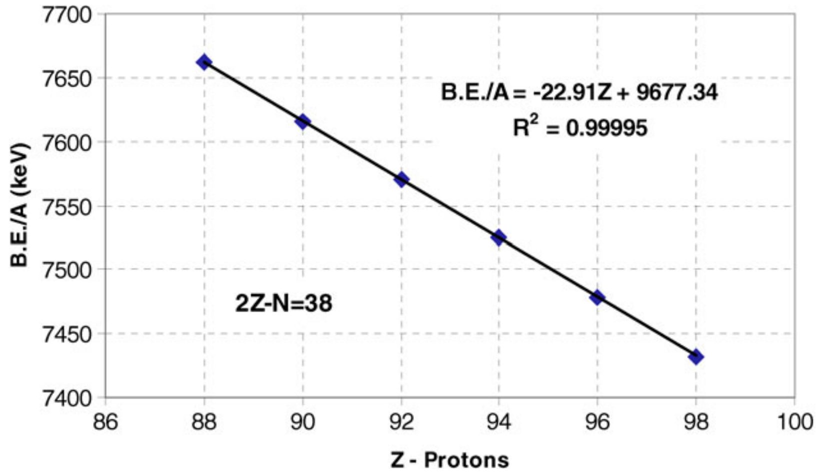

Fig. 3. The *B.E./A* with respect to the atomic number Z for heavy even isotopes $88 \leq Z \leq 98$ with $2Z - N = 38$.

TABLE I
Even(Z)-Even(N) Isotopes

$2Z - N$	$-a$ [Eq. (1)]	b [Eq. (1)]	Correlation Factor R^2	Number of Isotopes
36	22.94	9663.26	0.99778	4
38	22.91	9677.34	0.99995	6
40	22.67	9671.92	0.99996	5
42	22.43	9664.16	0.99995	5
44	22.12	9647.06	0.99997	5
46	22.11	9656.46	0.99983	6
48	21.86	9640.78	0.99944	6
50	21.92	9652.76	0.99904	7
52	21.86	9651.41	0.99780	6
54	22.26	9692.82	0.99552	5
Total				55

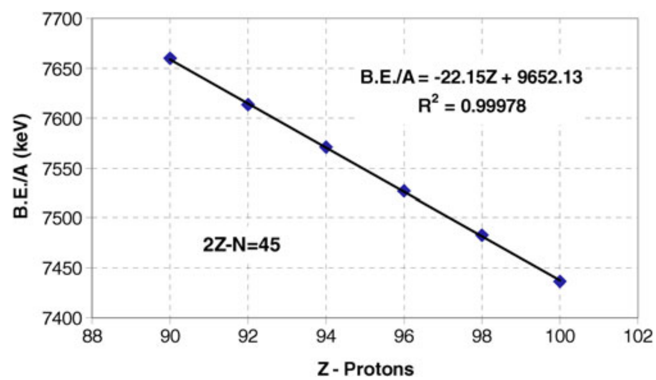

Fig. 4. The *B.E./A* with respect to the atomic number Z for heavy even isotopes $90 \leq Z \leq 100$ with $2Z - N = 45$.

In this analysis, we have considered 55 even-even isotopes, 44 even-odd isotopes, 32 odd-even isotopes, and 29 odd-odd isotopes. The total number of isotopes considered is 160. We have found very good linear correlations $R^2 > 0.99$ between the *B.E./A* with respect to Z as presented in Tables I through IV. Altogether, we have 31 different correlations, and the parameter $-a$ [in Eq. (1)] is similar in all the cases considered, with an average of -22.28 and a standard deviation of 0.30.

So, we have 160 heavy isotopes ($Z \geq 88$) with

$$\frac{\partial(B.E./A)}{\partial Z} = -22.28 \pm 0.30 \text{ (keV)} \ . \tag{2}$$

From Tables I through IV, we see that there is dependence of the parameter b [in Eq. (1)] with respect to the $2Z - N$ values. Dependence for the even(Z)-even(N), even(Z)-odd(N), odd(Z)-even(N), and odd(Z)-odd(N) is presented in Figs. 7 through 10. We see the linear dependence of the parameter b of Eq. (1) with respect to the $2Z - N$ values, namely,

$$b = c(2Z - N) + d \ . \tag{3}$$

In even(Z)-odd(N) and even(Z)-even(N) cases, the parameter c of Eq. (3) is negative, while it is positive for odd(Z)-even(N) and odd(Z)-odd(N). Even(Z)-odd(N) isotopes and odd(Z)-even(N) do not behave in the same way. Therefore, we considered odd-even and even-odd isotopes separately. We also found that parameter c is positive or negative as determined by whether Z is odd or even.

The excellent correlations obtained for the *B.E./A* are used in order to predict the *B.E./A* for isotopes without experimental values. We can use these correlations in order to predict the *B.E./A* for nuclei within the correlated isotopes and interpolated values or to predict the *B.E./A* for nuclei outside the correlated isotopes and extrapolated values. In our examples, we studied interpolated values, which are more reliable. One

TABLE II
Even(Z)-Odd(N) Isotopes

$2Z-N$	$-a$ [Eq. (1)]	b [Eq. (1)]	Correlation Factor R^2	Number of Isotopes
37	22.62	9639.31	0.99991	5
39	22.44	9639.97	0.99995	4
41	22.43	9654.31	0.99981	5
43	22.445	9668.96	0.99965	6
45	22.15	9652.13	0.99978	6
47	21.78	9628.31	0.99966	6
49	21.67	9622.97	0.99921	6
51	21.79	9638.28	0.99808	6
Total				44

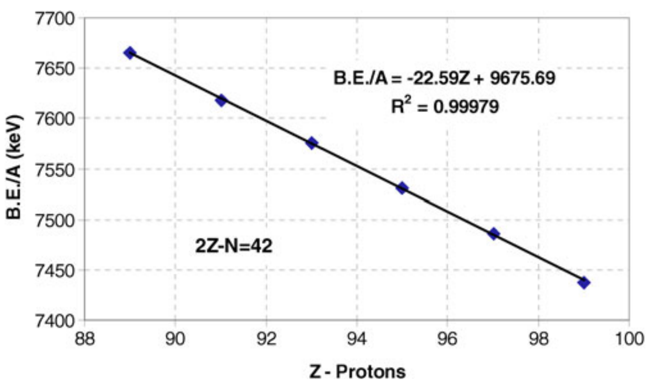

Fig. 5. The $B.E./A$ with respect to the atomic number Z for heavy odd isotopes $89 \leq Z \leq 99$ with $2Z-N=42$.

TABLE III
Odd(Z)-Even(N) Isotopes

$2Z-N$	$-a$ [Eq. (1)]	b [Eq. (1)]	Correlation Factor R^2	Number of Isotopes
38	22.18	9606.64	0.999976	3
40	22.62	9664.02	0.999964	5
42	22.59	9675.69	0.999786	6
44	22.40	9670.68	0.999668	6
46	22.38	9678.94	0.999456	7
48	22.31	9680.92	0.998696	5
Total				32

interpolated isotope with no experimental value that was considered was $^{232}_{93}$Np with $2Z-N=47$. The calculated $B.E./A$ from this correlation is 7600.0 (keV); the standard calculated result[1] is 7597 keV. We can also apply the value $\partial B.E./A = -22.28$ (keV) from Eq. (2) to the closest isotopes that have

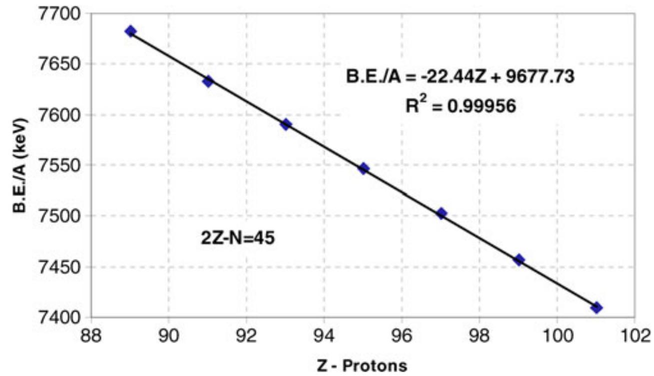

Fig. 6. The $B.E./A$ with respect to the atomic number Z for heavy odd isotopes $89 \leq Z \leq 101$ with $2Z-N=45$.

TABLE IV
Odd(Z)-Odd(N) Isotopes

$2Z-N$	$-a$ [Eq. (1)]	b [Eq. (1)]	Correlation Factor R^2	Number of Isotopes
37	22.033	9582.4	0.999998	3
39	22.354	9628.9	0.999986	4
41	22.380	9647.1	0.999923	5
43	22.514	9672.8	0.999770	5
45	22.441	9677.7	0.999560	7
47	22.419	9685.0	0.999511	5
Total				29

experimental $B.E./A$ values and the same $2Z-N=47$ value as $^{232}_{93}$Np, i.e., $^{226}_{91}$Pa and $^{238}_{95}$Am. The isotope $^{226}_{91}$Pa has a $B.E./A$ of 7641.11 keV; thus, using the value of $B.E./A = -22.28$ (keV) for the $B.E./A$ of $^{232}_{93}$Np, we obtain $7641.11 - 2 \cdot 22.28 = 7596.55$ keV. With $^{238}_{95}$Am, for the $B.E./A$ of $^{232}_{93}$Np, we obtain $7555.58 + 2 \cdot 22.28 = 7600.1$ keV. The presented correlations yield three values for the $B.E./A$ of $^{232}_{93}$Np. These values are 7600.0, 7596.4, and 7600.1 keV, compared to the standard calculated[1] value of 7597 keV. In another example, we examine the $B.E./A$ of $^{237}_{95}$Am resulting from the correlation with $2Z-N=48$, and we obtain a $B.E./A$ value of 7561 (keV). The calculated value[1] is also 7561 keV. For $^{236}_{96}$Cm with $2Z-N=52$, the calculated $B.E./A$ from the correlation yields a value of 7552.9 keV, and the standard calculated[1] value is 7550 ± 1 keV. For $^{240}_{98}$Cf with $2Z-N=54$, both the calculated[1] value and the one obtained from the correlation are the same, namely, 7510 keV. The last example is $^{248}_{97}$Bk, with a standard calculated[1] value of 7491 (keV) and a correlated value of 7489 keV.

From these comparisons, we see that the differences between the calculated results[1] and those obtained from the present correlations are at most 3 keV.

In order to gain better insight into the linear dependence of the $B.E./A$ with respect to the atomic number Z, we examined the $B.E./A$ dependence on Z as derived from the Liquid-Drop Model. For this analysis, we considered isotopes with $50 \leq Z \leq 124$, which includes all the actinides found in our

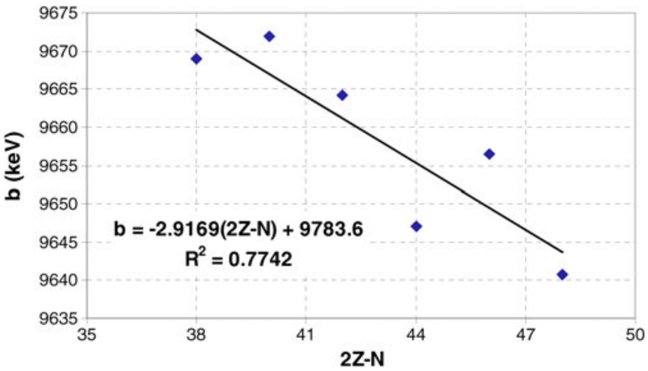

Fig. 7. The parameter b of Eq. (1) versus $2Z - N$ for even(Z)-even(N) heavy isotopes.

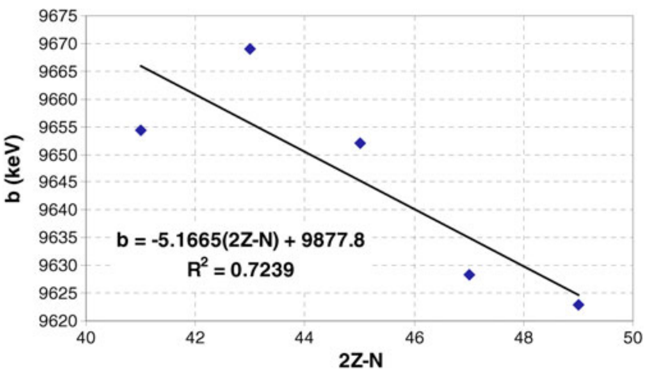

Fig. 8. The parameter b of Eq. (1) versus $2Z - N$ for even(Z)-odd(N) heavy isotopes.

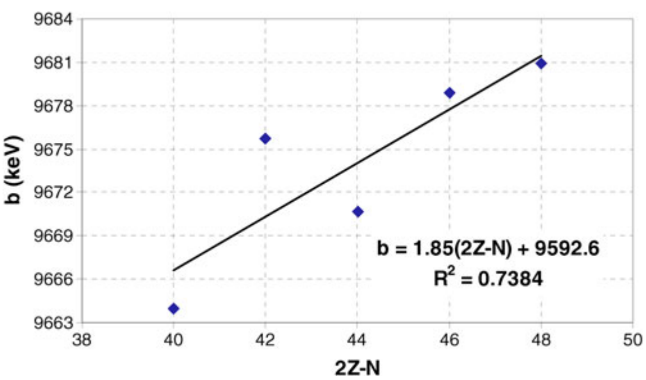

Fig. 9. The parameter b of Eq. (1) versus $2Z - N$ for odd(Z)-even(N) heavy isotopes.

above correlations. We have grouped these isotopes into even(Z)-even(N), odd(Z)-odd(N), even(Z)-odd(N), and odd(Z)-even(N). When considering isotopes in these groups, there are only three terms in the Liquid-Drop Model that affect the $B.E./A$ and are dependent on Z. These terms are Coulomb

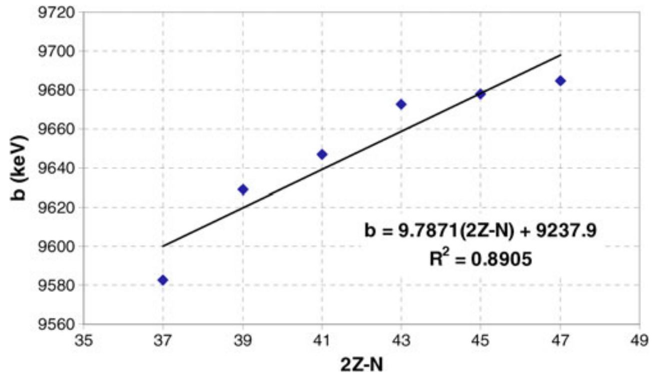

Fig. 10. The parameter b of Eq. (1) versus $2Z - N$ for odd(Z)-odd(N) heavy isotopes.

binding energy, surface energy, and symmetry energy, as presented in Eqs. (4), (5), and (6):

$$Coulomb\ B.E./A = -a_C \frac{Z^2}{A^{4/3}}\ (\mathrm{keV})\ , \tag{4}$$

where[2] $a_C = 697$ keV;

$$Surface\ B.E./A = -a_S A^{-1/3}\ (\mathrm{keV})\ , \tag{5}$$

where[2] $a_S = 17230$ keV;

$$Symmetry\ B.E./A = -a_a \frac{(N-Z)^2}{A^2}\ (\mathrm{keV})\ , \tag{6}$$

where[2] $a_a = 23285$ keV.

Earlier, we studied the dependence of the experimental $B.E./A$ on Z for isotopes that have the same $2Z - N$ value, namely, $2Z - N = \theta$, where θ is a constant that is an integer between 36 and 54 (see Table I). As a result, $N = 2Z - \theta$, and $A = 3Z - \theta$. These relations for A were introduced into Eqs. (4), (5), and (6).

The dependence of the Coulomb binding energy with respect to Z for different θ values for even-even isotopes is given in Fig. 11. We see the linear dependence of the Coulomb binding energy to the $B.E./A$ [Eq. (4)] with respect to Z, namely,

$$B.E./A = f \cdot Z + g\ . \tag{7}$$

There are very high correlations, $R^2 \geq 0.998$ for $\theta = 36$ up to $\theta = 54$, which include most of the actinides. We have also found that the values f and g of Eq. (7) are linearly dependent on $2Z - N$, as presented in Figs. 12 and 13.

The dependence of the surface and symmetry energies on Z for $2Z - N = 38$ is given in Fig. 14 and is clearly not linear with respect to Z. However, when combining all three terms from Eqs. (4), (5), and (6), we do obtain a linear dependence, as given in Fig. 15.

We have found that the experimental values of the $B.E./A$ for heavy isotopes ($Z \geq 88$) with $2Z - N =$ constant are highly linearly correlated. These isotopes are characterized by the fact that any additional proton that adds Coulomb repulsion is

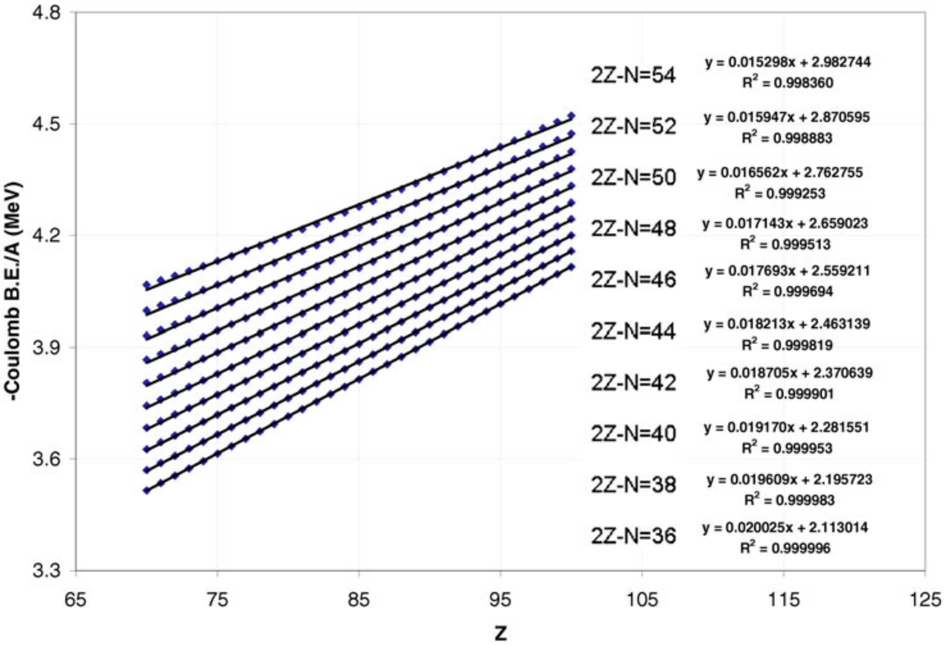

Fig. 11. The dependence of the Coulomb contribution to the *B.E./A* with respect to the atomic number ($70 \leq Z \leq 100$) for different $2Z - N$ values.

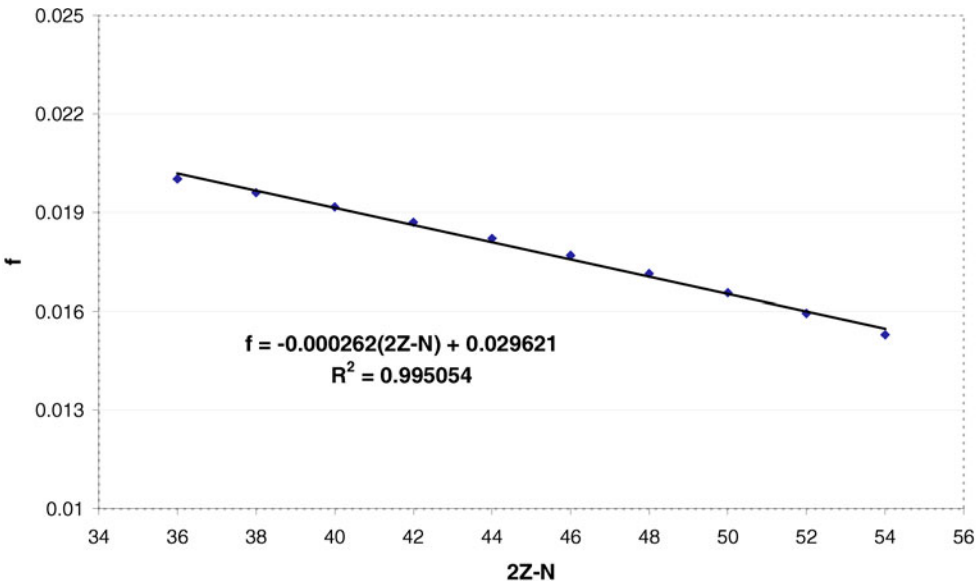

Fig. 12. The dependence of the parameter *f* of Eq. (7) with respect to $2Z - N$.

compensated by two neutrons that contribute to the attractive forces. The linear behavior of the *B.E./A* with respect to *Z* is also obtained in the Liquid-Drop Model. There are three terms in this model that influence the *B.E./A* and do not depend on the atomic number *Z*: the Coulomb binding energy, whose *B.E./A* contribution is linearly dependent on *Z*, and the symmetry and surface energies, whose *B.E./A* is not linearly correlated to *Z*. The nature of the contribution to the *B.E./A* of the three terms in the Liquid-Drop Model is determined by their mathematical forms. It is quite surprising that the addition of two nonlinear terms results in the linear dependence of the *B.E./A* with respect to *Z*. This is because these nonlinear terms are about equal but carry opposite signs ($0.1278Z^2$ and $-0.1578Z^2$). It was found[3] that many nuclear properties are

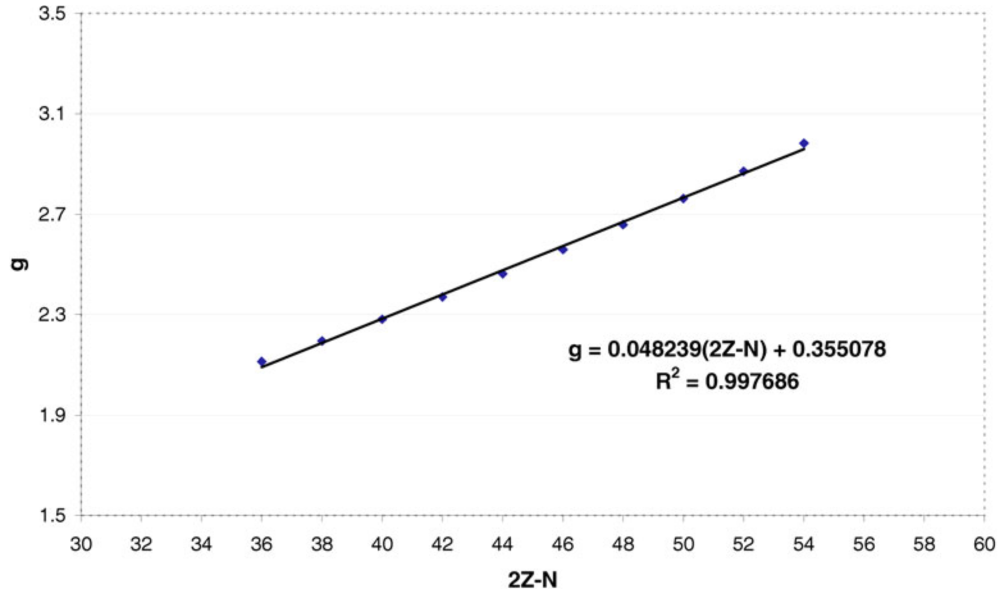

Fig. 13. The dependence of the parameter g of Eq. (7) with respect to $2Z - N$.

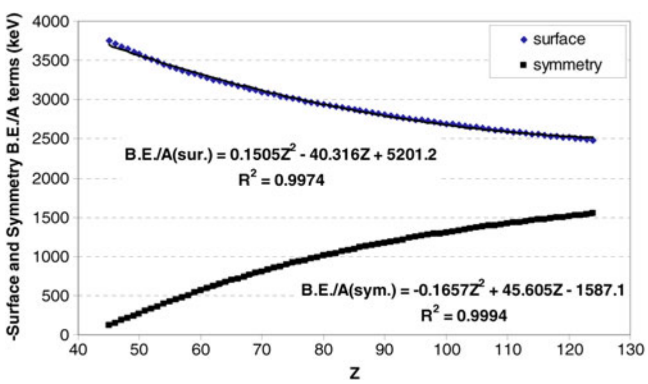

Fig. 14. The dependence of the surface and the symmetry contributions to the $B.E./A$ with respect to their atomic number ($45 \leq Z \leq 124$).

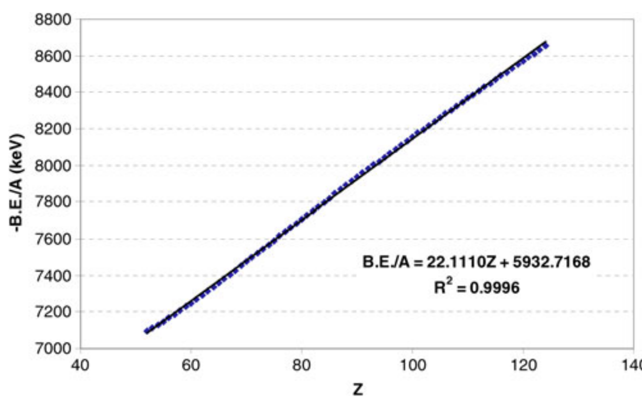

Fig. 15. The dependence of the contributions to the $B.E./A$ of the Coulomb, surface, and symmetry terms with respect to their atomic number ($50 \leq Z \leq 124$), for $2Z - N = 38$.

shared by isotopes that have the same $2Z - N$ values. In this respect, it was also recently found[4] that the behavior of the spontaneous branching ratios of even-Z isotopes is related to the $2Z - N$ values of these isotopes.

Another interesting observation is related to the fact that even(Z)-odd(N) isotopes behave differently than odd(Z)-even(N) isotopes in regard to the linear behavior of the $B.E./A$ with respect to Z. It was found that the b value of Eq. (1) is linearly dependent on $2Z - N$ [Eq. (3)]. The behavior of b is different for even(Z)-odd(N) isotopes, compared to odd(Z)-even(N) isotopes. Furthermore, the behavior of b with respect to $2Z - N$ values is determined only by the protons, namely, even(Z) or odd(Z). Based on the contribution of the pairing term to the $B.E.$, there should be no difference between even-odd and odd-even isotopes. It was also found that $\partial(B.E./A)/\partial Z = -22.28 \pm 0.30$ (keV), which is valid for 160 actinide isotopes without experimental values for their $B.E./A$.

We have found that the experimental values of the $B.E./A$ for high Z ($Z \geq 88$) isotopes are highly correlated to Z, when the isotopes are grouped into $2Z - N$ = constant. These high correlations serve to calculate the $B.E./A$ for nuclei that do not have experimental values and are used to obtain the $B.E./A$ for $^{232}_{93}$Np with a value of 7600.0 keV, $^{236}_{96}$Cm with a value of 7552.9 keV, $^{240}_{98}$Cf with a value of 7510.0 keV, and $^{248}_{97}$Bk with a value of 7489.0 keV. The results obtained from these correlations are compared to Liquid-Drop Model calculations, and the difference is, at most, 3 keV for these isotopes.

The linear dependence of the $B.E./A$ with respect to Z for $2Z - N$ = constant leads to the relation $\partial(B.E./A)/\partial Z = -22.28 \pm 0.30$ (keV). So the $(B.E./A)_2$, for an isotope with Z_2, is related to the $(B.E./A)_1$ of an isotope with Z_1 without an experimental value for its $B.E./A$, namely,

$$(B.E./A)_2 = (B.E./A)_1 + (Z_2 - Z_1)\,\frac{\partial(B.E./A)}{\partial Z} \;, \tag{8}$$

where $2Z - N$ = constant, also avoiding pairing effects. We should choose $(Z_2 - Z_1) = 2$ to yield $N_2 - N_1 = 4$. If we know the experimental value $(B.E./A)_1$, we can calculate the $(B.E./A)_2$ by $(B.E./A)_2 = (B.E./A)_1 + 44.56$ keV. The uncertainty for the value of $(B.E./A)_2$ is 0.6 keV, as long as the experimental uncertainty of $(B.E./A)_1$ is much less than 0.6 keV.

We do not have a satisfactory explanation for three results obtained in this study. We do not know why adding the contributions of the $B.E.$ of three terms in the Liquid-Drop Model (i.e., the Coulomb, symmetry, and surface energies) yields a linear dependence with respect to Z. The second unexplained result is that the experimental $B.E./A$ is linearly dependent on Z for high-Z isotopes. The third unexplained result is that even(Z)-odd(N) isotopes behave differently in some aspects related to their binding energy than odd(Z)-even(N) isotopes.

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