LETTERS TO THE EDITOR

COMMENTS ON "ELECTRON TRANSITIONS ON DEEP DIRAC LEVELS I"

In a recent paper,[1] the authors claim the existence of deeply bound electron energy levels in hydrogen-like atoms resulting from previously neglected solutions of both the relativistic Schrödinger and Dirac equations. In this letter, we show that these solutions are unphysical, and thus, these deeply bound energy levels cannot exist.

The radial part of the relativistic Schrödinger equation for a point Coulomb potential $e\phi(r) = -Ze^2/r$ is[2]

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\psi}{d\rho}\right) + \left[\frac{\lambda}{\rho} - \frac{1}{4} - \frac{l(l+1)-\gamma^2}{\rho^2}\right]\psi = 0 \ , \quad (1)$$

where

$$\rho = \alpha r \ , \quad \gamma = \frac{Ze^2}{\hbar c} \ , \quad \alpha^2 = \frac{4(m^2c^4 - E^2)}{\hbar^2c^2} \ , \quad \lambda = \frac{2E\gamma}{\hbar c\alpha} \ , \quad (2)$$

and E and m are the total energy and mass of the electron, respectively. We consider solutions of this equation for $l = 0$ only because the energies corresponding to $l > 0$ must be higher than the lowest $l = 0$ energy because of the centrifugal barrier. It is easily seen that as $\rho \to 0$, the wave function ψ has the behavior

$$\psi \sim \rho^{s_\pm} \ , \quad (3)$$

where

$$s_\pm = -\tfrac{1}{2} \pm (\tfrac{1}{4} - \gamma^2)^{1/2} \ . \quad (4)$$

From Eq. (2),

$$E = mc^2\left(1 + \frac{\gamma^2}{\lambda^2}\right)^{-1/2} \ , \quad (5)$$

and the boundary condition $\psi(\rho \to \infty) = 0$ gives $\lambda_\pm = n' + s_\pm + 1$, where n' is a positive integer. Furthermore, because $\gamma = Ze^2/\hbar c \approx Z/137 \ll 1$,

$$s_\pm \approx -\tfrac{1}{2} \pm (\tfrac{1}{2} - \gamma^2) \ , \quad (6)$$

the energy levels E_+ and E_- are given by

$$E_+ \approx mc^2\left[1 + \frac{\gamma^2}{(n' + 1 - \gamma^2)^2}\right]^{-1/2} \quad (7)$$

and

$$E_- \approx mc^2\left[1 + \frac{\gamma^2}{(n' + \gamma^2)^2}\right]^{-1/2} \ . \quad (8)$$

Note that Eq. (7) gives the correct observed binding energy of $E_+ - mc^2 = -13.6$ eV for $n' = 0$ and $Z = 1$, and Eq. (8) gives a binding energy of $E_- - mc^2 \approx mc^2(\gamma - 1) \approx -510$ keV, so that if the solutions corresponding to s_- are acceptable, there exist deeply bound electron states.

Our solutions, Eqs. (7) and (8), for the relativistic Schrödinger equation, Eq. (1), are similar to that of Maly and Vávra[1] [their Eq. (24)] for the Dirac equation.[2] The same shortcoming, detailed below, applies to their solution; however, it is less transparent than our example because the Dirac equation involves a set of coupled differential equations.[2]

We can model a realistic situation by assuming that the potential $e\phi(r)$ is given by

$$e\phi(r) = \begin{cases} -\dfrac{Ze^2}{a} \ , & r < a \\ -\dfrac{Ze^2}{r} \ , & r \geq a \ , \end{cases} \quad (9)$$

where a is the nuclear radius. The wave function $\chi(r) = r\psi(r)$ then has the following form, as $a \to 0$:

$$\chi(r) = \begin{cases} AKr \ , & r < a \\ B\rho^{s_+ + 1} + C\rho^{s_- + 1} \ , & r \gtrsim a \ , \end{cases} \quad (10)$$

where $(\hbar c)^2K^2 = (E + Ze^2/a)^2 - m^2c^4$. Note that in Eq. (10), we have used the regular solution $\chi = A \sin Kr$ for the interior wave function, which is zero at the origin, as it must be for a finite potential, and the form of the exterior wave function comes from the analytic solution of Eq. (1).

Equating logarithmic derivatives

$$\frac{1}{\chi}\frac{d\chi}{dr}\bigg|_{r=a^-} = \frac{1}{\chi}\frac{d\chi}{dr}\bigg|_{r=a^+} \quad (11)$$

gives

$$\frac{C}{B} = -\frac{s_+}{s_-}(\alpha a)^{s_+ - s_-} \approx -\frac{\gamma^2}{1-\gamma^2}(\alpha a)^{1-2\gamma^2} \ , \quad (12)$$

so that for a physical wave function, as $a \to 0$, the solution corresponding to s_- does not contribute. However, because of the finite size of the nucleus, the wave function consists of a large component corresponding to s_+ and a small component corresponding to s_- with a binding energy that is thus very close to the original binding energy $E_+ - mc^2$. For instance, for the case of a proton, the proton radius $a \approx 1$ fm, $Z = 1$, and $E \approx E_+ = mc^2 - 13.6$ eV; therefore, $\alpha \approx 3.78 \times 10^{-5}$ fm^{-1}, and hence, $C/B \approx -0.2 \times 10^{-8}$.

Furthermore, we note that their electron orbits of radius $\sim 5 \times 10^{-13}$ cm are 50 times smaller than muonic orbits of 250×10^{-13} cm. If such orbits existed, upon collision, they would produce fusion at a much higher rate than muon-catalyzed fusion. So, their proposed solution does not get around the nuclear ash problem.

In summary, we have shown that the deep Dirac orbits do not exist. They are an artifact due to the incorrect use of the irregular solution of the relativistic Dirac equation and the total neglect of the regular solution. Furthermore, even if such orbits existed, this does not keep the problem in the domain of exotic chemistry with the avoidance of nuclear effects and the nuclear ash problem. Such tight orbits would be expected to produce considerably high fusion rates.

R. A. Rice
Y. E. Kim

Purdue University
Department of Physics
West Lafayette, Indiana 47907

Mario Rabinowitz

Electric Power Research Institute
Palo Alto, California 94303

September 3, 1993

REFERENCES

1. J. A. MALY and J. VÁVRA, "Electron Transitions on Deep Dirac Levels I," *Fusion Technol.*, **24**, 307 (1993).

2. L. I. SCHIFF, *Quantum Mechanics*, 3rd ed., McGraw-Hill Book Company, New York (1968).

RESPONSE TO "COMMENTS ON 'ELECTRON TRANSITIONS ON DEEP DIRAC LEVELS I'"

The central point of our response to Ref. 1 is as follows. Let us assume that $\psi_n(x)$ is a solution of the time-independent portion of the Schroedinger equation for energy E_n. The authors of Ref. 1 indicate that if $\psi_n(x)$ and $\psi_m(x)$ are solutions, then $A\psi_n(x) + B\psi_m(x)$ is also the solution, thus completely disregarding the fact that n and m are two different quantum states with different energies E_n and E_m; $\psi_n(x)$ and $\psi_m(x)$ are the solutions of two different differential equations. The complete solution of the time-dependent Schroedinger equation for a particle in the quantum state $\psi_n(x)$ of energy E_n is

$$\psi_n(x,t) = \exp[-i(E_n/\hbar)t]\psi_n(x) \ . \tag{1}$$

The probability distribution $\psi^* \cdot \psi$ is independent of time; thus, the particle stays in the energy state E_n indefinitely. Therefore, it is not a portion of time in the energy state E_m, as the authors indicate when they write the wave function $[A\psi_n(x) + B\psi_m(x)]$.

Specifically, our response to their criticism is the following:

1. The authors[1] combine in a linear combination two solutions corresponding to s^+ and s^- parameters. They correspond to two different quantum states, which, in turn, correspond to two different differential equations and two different energy levels. To use the linear combination $\psi = B\rho^{s^+ + 1} + C\rho^{s^- + 1}$ as a general wave function would be correct only if both terms $B\rho^{s^+ + 1}$ and $C\rho^{s^- + 1}$ would be the solution of the same differential equation. However, this is not the case. The equation, which has to be solved, is Eq. (9) in Ref. 2:

$$\rho^2 L'' + \rho[2(s+1) - \rho]L' + [\rho(\lambda - s - 1) + s(s+1) - l(l+1) + \gamma^2]L = 0 \ . \tag{2}$$

This equation for $\rho = 0$ is zero only if it is satisfied:

$$s(s+1) + \gamma^2 - l(l+1) = 0 \ , \tag{3}$$

which gives the solution [see Schiff[3] Eq. (51.19) or Eq. (10) in Ref. 2]:

$$s = -\tfrac{1}{2} \pm [(l + \tfrac{1}{2})^2 - \gamma^2]^{1/2} \ . \tag{4}$$

The aforementioned s^+ and s^- correspond to two different signs in Eq. (4). Because s^+ and s^- are clearly different, Eq. (2) with inserted $s = s^+$ is different from Eq. (2) with inserted $s = s^-$; the resulting energy levels E_{s^+} or E_{s^-} are different for s^+ or s^- [see Eqs. (12) and (5) of Ref. 2]. Thus, one cannot use $\psi = B\rho^{s^+ + 1} + C\rho^{s^- + 1}$ as the most general wave function [$B\rho^{s^+ + 1}$ contains E_{s^+} and $C\rho^{s^- + 1}$ contains E_{s^-} inside $\rho = \alpha(E)r$].

2. Equations (9) of Ref. 1 define the potential inside and outside the nucleus:

$$e\phi(r) = \begin{cases} -Ze^2/a & \text{for } r \le a \\ -Ze^2/r & \text{for } r > a \ . \end{cases} \tag{5}$$

The potential outside the nucleus is correct, but the potential inside the nucleus is only an approximation. However, two errors appear in Eq. (10) of Ref. 1:

$$\chi(r) = rR(r) = \begin{cases} AKr & \text{for } r \le a \\ B\rho^{s^+ + 1} + C\rho^{s^- + 1} & \text{for } r > a \ , \end{cases} \tag{6}$$

where $\hbar^2 c^2 K^2 = (E + Ze^2/a)^2 - m^2 c^4$ and $l = 0$. First, the correct solution of the differential equation for $r \le a$ is $\chi(r) = A \sin(Kr)$ and not AKr, as used in Eqs. (10) and (12) of Ref. 1 [this can be derived from Eq. (51.14) of Schiff[3] by using $l = 0$ and the potential of Eq. (5)]. Second, as we explained earlier, for $r > a$, the solutions are either $\chi(r) = C_1 r R_1(\rho)$ for $s = s^+$ or $\chi(r) = C_2 r R_2(\rho)$ for $s = s^-$ [$R(\rho)$ is defined by Eqs. (7) and (8) of Ref. 2] and not a linear combination of these two solutions as shown in Eq. (6).

3. The authors of Ref. 1 have also not realized that the variable $\rho = \alpha(E)r$ contains energy levels E_{s^+} or E_{s^-} inside parameter $\alpha(E)$ [see Eq. (5) of Ref. 2], and ρ in Eq. (6) is not the same for s^+ or s^- states.