Comments on "Exact Time-Dependent Second Spatial Moment of the One-Speed Neutron Transport Model"

In a recent Note, Barnett[1] presented a calculation of the time-dependent second moment of the one-velocity neutron distribution. I would like to point out that if one is interested in the *moments*, it is unnecessary to calculate the full distribution; instead, one should start directly with equations for the moments. The use of moments is not new; they have been applied quite often to the age equation in which the age is the analog of the time variable. In general, the age equation is more complicated because in that case the coefficients of the differential equation are age (time) dependent.

We begin with the one-dimensional transport equation

$$\frac{\partial\phi}{\partial t} + \mu\frac{\partial\phi}{\partial x} + (1+a)\,\phi(x,\mu,t) = \int_{-1}^{1}\phi(x,\mu',t)\,P(\mu,\mu')\,d\mu' + S(x,\mu,t) \quad , \tag{1}$$

where we have set the velocity and the scattering mean free path to unity. In these units, the *total* cross section is $1 + a$, where a is the ratio of adsorption to scattering (a is assumed constant). If the scattering function $P(\mu, \mu')$ is a function only of the angle of scattering, θ, with

$$\cos\theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos\phi \quad , \tag{2a}$$

then

$$P(\mu,\mu') = \int P(\theta)\,d\phi \quad , \tag{2b}$$

and if $P(\theta)$ can be expanded in Legendre polynomials,

$$P(\theta) = \frac{1}{4\pi}\,\Sigma(2n+1)\,c_n P_n(\cos\theta) \quad , \quad c_0 = 1, \quad c_1 = \bar{\mu} \quad , \tag{2c}$$

one has immediately

$$P(\mu,\mu') = \tfrac{1}{2}\,\Sigma(2n+1)\,c_n P_n(\mu)\,P_n(\mu') \quad . \tag{2d}$$

If one now defines the harmonic moments $G_{mk}(t)$,

$$G_{mk}(t) = \int_{-1}^{1} P_k(\mu)\int_{-\infty}^{\infty} x^m\phi(x,\mu,t)\,dx\,d\mu \quad ,$$

with a similar equation for S_{mk}, the moments of the source, Eq. (1) is transformed into

$$\frac{dG_{mk}}{dt} + (1+a-c_k)\,G_{mk}(t) = \frac{m}{2k+1}\left[kG_{m-1,k-1}(t) + (k+1)G_{m-1,k+1}(t)\right] + S_{mk}(t) \quad , \tag{3}$$

which is an ordinary differential equation for the m'th moments recursively in terms of the $(m-1)$'th moments.

For the steady-state moments, one sets

$$S(x,\mu,t) = \frac{a}{2}\,\delta(x)$$

and omits the time dependence in Eqs. (1) and (3). The source moments are then $S_{mk} = \delta_{m0}\delta_{k0}$, and the moments can be computed by straightforward recursion. One can characterize the moments by the total order $p = m + k$, and each moment may be computed from moments of lower order in the sequence 00;11,20;22,31,40;33,42,51,60; . . . . With the given source, all other moments vanish. Then

$$G_{00} = 1 \tag{4a}$$

$$G_{20} = \frac{1}{3a(1+a-\bar{\mu})} \tag{4b}$$

$$G_{40} = \frac{8(5+9a-5c_2)}{15a^2(1+a-\bar{\mu})^2(1+a-c_2)} \quad . \tag{4c}$$

For the time-dependent moments,

$$S(x,\mu,t) = \tfrac{1}{2}\,\delta(x)\,\delta(t)$$

$$S_{mk} = \delta_{m0}\delta_{k0}\delta(t) \quad ,$$

and one finds

$$\frac{dG_{00}(t)}{dt} + aG_{00}(t) = \delta(t) \quad \text{or} \quad G_{00}(t) = \exp(-at) \quad , \tag{5}$$

so that the total neutron field decays exponentially.

The first-order moments $G_{01}(t)$ and $G_{10}(t)$ both vanish. The nonvanishing second-order moments are given by

$$\frac{dG_{02}(t)}{dt} + (1+a-c_2)\,G_{02}(t) = 0 \tag{6a}$$

$$\frac{dG_{11}(t)}{dt} + (1+a-\bar{\mu})\,G_{11}(t) = \frac{1}{3}\left[G_{00}(t) + 2G_{02}(t)\right] \tag{6b}$$

$$\frac{dG_{20}(t)}{dt} + aG_{20} = 2\,G_{11}(t) \quad . \tag{6c}$$

Integration of these equations leads to

$$G_{02}(t) = 0 \tag{7a}$$

$$G_{11}(t) = \frac{\exp(-at)}{3(1-\bar{\mu})}\,\{1 - \exp[-(1-\bar{\mu})\,t]\} \tag{7b}$$

$$G_{20}(t) = \frac{2\exp(-at)}{3(1-\bar{\mu})}\left\{t - \frac{1-\exp[-(1-\bar{\mu})\,t]}{1-\bar{\mu}}\right\} \quad . \tag{7c}$$

Equation (7c) then leads directly to Barnett's result. This procedure is significantly shorter than Barnett's derivation for the second moment, but the real advantage comes when one attempts to compute higher moments.

[1] C. S. BARNETT, *Nucl. Sci. Eng.*, **50**, 398 (1973).

For the fourth-order moments, we write

$$\frac{dG_{22}}{dt} + (1 + a - c_2)\, G_{22}(t) = \frac{4}{5}\, G_{11}(t) \tag{8a}$$

$$\frac{dG_{31}}{dt} + (1 + a - \bar{\mu})\, G_{31}(t) = G_{20}(t) + 2\, G_{22}(t) \tag{8b}$$

$$\frac{dG_{40}}{dt} + aG_{40}(t) = 4\, G_{31}(t) \quad . \tag{8c}$$

The integrations are straightforward, although increasingly tedious:

$$G_{22}(t) = \frac{4 \exp(-at)}{15(1 - \bar{\mu})} \left\{ \frac{1 - \exp[-(1 - c_2)t]}{1 - c_2} - \frac{\exp[-(1 - \bar{\mu})t] - \exp[-(1 - c_2)t]}{\bar{\mu} - c_2} \right\} \quad , \tag{9a}$$

$$G_{31}(t) = \frac{2 \exp(-at)}{3} \left(\frac{1}{(1 - \bar{\mu})^3} \{[2 + (1 - \bar{\mu})t] \exp[-(1 - \bar{\mu})t] - 2 + (1 - \bar{\mu})t\} + \frac{4}{5} \frac{\{1 - \exp[-(1 - \bar{\mu})t]\}}{(1 - c_2)(1 - \bar{\mu})^2} - \frac{4t \exp[-(1 - \bar{\mu})t]}{5(1 - \bar{\mu})(\bar{\mu} - c_2)} \right.$$
$$\left. + \frac{4}{5(1 - c_2)} \left\{ \frac{\exp[-(1 - \mu)t] - \exp[-(1 - c_2)t]}{(\mu - c_2)^2} \right\} \right) \quad , \tag{9b}$$

and

$$G_{40}(t) = \frac{8 \exp(-at)}{3} \left[\frac{3 - 2\xi + \frac{1}{2}\xi^2 - (3 + \xi)\exp(-\xi)}{(1 - \bar{\mu})^4} + \frac{4}{5(1 - \bar{\mu})^3} \left[\frac{\exp(-\xi) - 1 + \xi}{1 - c_2} - \frac{1 - (1 + \xi)\exp(-\xi)}{\bar{\mu} - c_2} \right] \right.$$
$$\left. + \frac{4\big((1 - c_2)[1 - \exp(-\xi)] - (1 - \bar{\mu})\{1 - \exp[-(1 - c_2)t]\}\big)}{5(1 - c_2)^2 (\bar{\mu} - c_2)^2 (1 - \bar{\mu})} \right] \quad , \tag{9c}$$

where $\xi \equiv (1 - \bar{\mu})\, t$.

The three-dimensional moments may be found most easily from the plane-point transformation.[2] The distribution from a point source $\Phi(r)$ in terms of the distribution from a plane source $\phi(x)$ is

$$\phi(r) = -\frac{1}{2\pi r}\, \phi'(r) \quad , \tag{10}$$

and the radial moments,

$$M_{2n}(t) = \int \phi(r) r^{2n} \cdot 4\pi r^2\, dr \quad ,$$

are simply

$$M_{2n}(t) = (2n + 1)\, G_{2n,0}(t) \quad .$$

Then, for short times, $t \ll 1$, we have $M_4(t) \to t^4$, and for $t \gg 1$, the distribution of the unabsorbed neutrons is

$$\langle R^4(t) \rangle \to \frac{20}{3(1 - \bar{\mu})^2} \times \left[t^2 - \frac{4t}{1 - \bar{\mu}} + \frac{6}{(1 - \bar{\mu})^2} + \frac{8}{5(1 - c_2)} \left(t - 1 - \frac{1 - \bar{\mu}}{1 - c_2} \right) \right] . \tag{11}$$

The mean fourth moment to absorption can be calculated from Eq. (9c) and is given by

$$\langle R^4 \rangle = \int_0^\infty 5\, G_{40}(t)\, a\, dt = \frac{8[5(1 - c_2) + 9a]}{3a^2(1 - \mu + a)^2 (1 - d_2 + a)} \quad , \tag{12}$$

which agrees with Eq. (4c).

The higher moments can be similarly calculated, but there is probably no need for exhibiting the results explicitly; the recursive formulas are easily converted to numerical calculation.

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[2]B. DAVISON and J. B. SYKES, *Neutron Transport Theory*, p. 64, Oxford University Press (1958).

Reply to "Comments on 'Exact Time-Dependent Second Spatial Moment of the One-Speed Neutron Transport Model' "

Cohen's approach to calculating the time-dependent moments by starting with the transport equation is elegant and efficient. I suppose that the simplicity of the second moment formula should lead one to suspect that it must be the solution to some simple differential equation. This, in turn, should suggest that closed systems of moments equations might be obtained—a rare and rewarding occurrence.

Cohen mentions that "it is unnecessary to calculate the full distribution" to find spatial moments. I assume he means, speaking probabilistically, that it is unnecessary to find the probability density function that describes the neutron's distance from the origin at time t; or, in transport theory terms, that it is unnecessary to find an expression for the neutron number density as a function of radius and time. He is surely correct, and I certainly did not find, or attempt to find, the probability density function in question. To do so would mean that the entire point-source, time-dependent problem would be solved. In effect, Case and Zweifel[1] have solved the complete problem for the infinite one-dimensional case. They find the Green's function for the monodirectional plane source of neutrons at time zero problem. Their solution is somewhat formal and involves integrals of complex valued function.

For those who are facile with the transport equation, several comments and questions might be of interest:

1. Is it possible to extract the time-dependent spatial moments from the Case and Zweifel solution?
2. Can the transport equation approach shed any light on the odd time-dependent spatial moments? The odd moments are zero in the one-dimensional case but

[1]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, pp. 186-187, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).