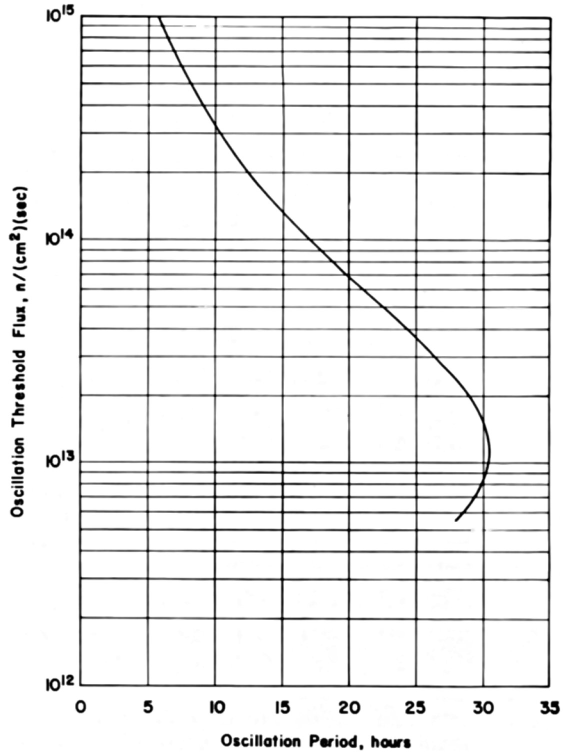

FIG. 4. Period of xenon oscillation as a function of the threshold value of the neutron flux.

tion of the radius that is flattened, for the case of a zero power coefficient. For comparison the corresponding curves for axial oscillations (*1*) are included in Figs. 2 and 3 as dashed lines, where, for a cylindrical reactor with a core diameter equal to the core height, the corresponding value of H^2/M^2 is four times that of R^2/M^2. We see that, for such a reactor, more buckling must be added to excite the first radial harmonic than to excite the first axial harmonic, so that the threshold flux for radial oscillation lies above that for an axial oscillation.

Figure 4 shows the oscillation period at the threshold as a function of the threshold flux level. As shown in Eq. (11) of ref. 1, this period is a function of the threshold flux only and not directly a function of the pile size, flatness, or oscillation mode. Thus the curve of Fig. 4 is good for any degree of flatness and for both axial and "tilt" oscillations.

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D. RANDALL
D. S. ST. JOHN

Savannah River Laboratory
E. I. du Pont de Nemours & Co.
Aiken, South Carolina

Received March 28, 1962
Revised May 18, 1962

The Transfer Cross Section Between Overlapping Thermal Groups

A neutron thermalization problem which has received some attention in the literature (*1*, *2*) is that of finding the variation in space of the thermal neutron spectrum in two adjoining regions of scattering material at different temperatures T and T_0. The overlapping thermal group theory (*2*) approaches this problem by assuming that the neutron flux spectrum has the form

$$\Phi(x, v) = \phi(x)M(v) + \phi_0(x)M_0(v) \tag{1}$$

where v is the velocity, x is the space coordinate, and M and M_0 are maxwellian distributions at the bulk temperatures T and T_0 of the two media. Then the thermalization problem becomes one of solving two diffusion equations in the neutron groups $\phi(x)$ and $\phi_0(x)$ which are considered monoenergetic for the purposes of diffusion. These equations have been given by Selengut (*2*) and are discussed at the end of the letter. In this model neutron thermalization is replaced by a transfer process which takes the neutrons from one group to the other. Selengut (*2*) has given a "transfer cross section"

$$\sigma_T = \frac{2}{A}\sigma_{fa} \tag{2}$$

for a heavy gas of mass A and free atom cross section σ_{fa}. This letter considers a free gas of any mass A, and derives an expression for σ_T, which is valid once assumption (1) has been made. For the part of the spectrum at temperature T_0 in the medium at temperature T it is

$$\sigma_T(A, T_0 \to T) = \frac{2}{A}\sigma_{fa}\frac{\sqrt{1 + T/(AT_0)}}{(1 + 1/A)^2}. \tag{3}$$

In the medium at temperature T, the first part of the spectrum in Eq. (1) does not exchange energy with the medium since it is also at temperature T. However the second part of the spectrum in Eq. (1) does exchange energy because it is at the different temperature T_0. In the overlapping group theory the second group is transferred to the first group in such a way that the rate of change of energy is made equal to that calculated in free gas theory. In a free gas at temperature T which has a scattering cross section σ_s, the rate of change of average energy per neutron of a neutron spectrum which is a maxwellian at temperature T_0 is

$$\frac{\overline{dE_0}}{dt} = -n\int \sigma_s(E_0)\langle E_0 - E\rangle_{av}\, v_0 M_0\, dv_0 \tag{4}$$

where n is the number of atoms per cubic centimeter, $\langle E_0 - E\rangle$ is the average energy change undergone by a neutron of energy E_0 in a collision in a free gas at temperature T, and

$$M_0 = \frac{4}{\sqrt{\pi}}\left(\frac{2kT_0}{m}\right)^{-3/2} v_0^2 \exp\left(\frac{-mv_0^2}{2kT_0}\right) \tag{5}$$

is the fraction of neutrons per unit velocity interval. von Dardel (*3*) has evaluated first the quantity $\langle E_0 - E\rangle_{av}$ and then the integral (*4*) explicitly in the course of his work on neutron cooling, obtaining

$$\frac{\overline{dE_0}}{dt} = -\frac{8}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{1/2} n\sigma_{fa}\frac{k(T_0 - T)}{A(1+1/A)^{3/2}} \cdot \left[1 + \frac{T_0 - T}{T(1+1/A)}\right]^{1/2} \tag{6}$$

where m is the neutron mass.

We now require that $\overline{dE_0}/dt$ from the gas model be equal to the average rate of energy change of the ϕ_0 group in the overlapping model, which is the product of the transfer rate from ϕ_0 to ϕ, $n\sigma_T\int v_0 M_0 dv_0$, and the change of energy per neutron transferred, $\bar{E} - \overline{E_0}$.

$$\frac{\overline{dE_0}}{dt} = -(\overline{E_0} - \bar{E})n\sigma_T \int v_0 M_0\, dv_0 \tag{7}$$

This procedure defines σ_T, but first $\overline{E_0} - \overline{E}$ must be chosen. In the steady state diffusion of the T_0 group in the medium at temperature T, the transfer to the group T is the sink term (in the case of no absorption) and the net inward diffusion of T_0 neutrons is the source term. Since the average energy per neutron diffusing into an elementary volume is $2kT_0$, the energy per neutron transferred must be $2kT_0$. Thus $\overline{E_0} - \bar{E} = 2k(T_0 - T)$. This is the expression used by Selengut (*2*) and is consistent with the assumption (1) that the shape M_0 is retained throughout the transfer process.

Then from Eqs. (6) and (7) and putting $\int v_0 M_0 dv_0 = \overline{v_0} = (2/\sqrt{\pi})\sqrt{2kT_0/m}$, we get the final expression for σ_T given in Eq. (3).

The expression (3) for $\sigma_T(A, T_0 \to T)$ satisfies three criteria which are expected to hold on physical grounds:

1. $\lim_{A\to 0} \sigma_T(A) = 0$ i.e., there should be vanishing energy transfer for a hypothetical free gas much lighter than the neutron.

2. σ_T should have a maximum value in the neighbourhood of $A = 1$ since equal mass particles give optimum energy transfer. For a theory in which the scattering cross section is not a function of energy, this maximum should occur exactly at $A = 1$. However it is well known at low energy in a free gas that σ_s increases with decreasing A because of increasing encounters due to thermal motion of the gas. Thus the maximum in σ_T should actually occur below $A = 1$. The maximum in $\sigma_T(A, T_0 \to T)$ in Eq. (3) in the case $T_0 \sim T$ occurs at $A = 1/2$.

3. In a practical problem, the heating of the gas atoms is always insignificant, but from a formal point of view the overlapping group result should be applicable not only to the heating of the neutrons by the gas, but also to the heating of the gas by the neutrons, and this leads to a certain symmetry in σ_T. Assume that the gas spectrum is also the sum of two overlapping groups, and consider a case where all the neutrons are initially at temperature T_0 while all the gas is at temperature T. The rate of heat loss by the neutrons must equal the rate of heat gain by the gas atoms. But for both neutrons and gas atoms, the energy difference between groups is $2kT - 2kT_0$ per particle transferred. Thus, the transfer rate of atoms between groups must equal the transfer rate of neutrons between groups. Equating these two transfer rates and cancelling the products of the gas density and atom density which occur in both, we obtain the criterion

$$\sqrt{T_0}\,\sigma_T(A, T_0 \to T) = \sqrt{\frac{T}{A}}\,\sigma_T\left(\frac{1}{A}, T \to T_0\right).$$

The factors $\sqrt{T/A}$ and $\sqrt{T_0}$ are simply proportional to the fluxes of gas atoms and of neutrons respectively. Criterion 3 can be seen to be satisfied by substituting Eq. (3) for σ_T.

The expression (2) for σ_T from heavy gas theory does not satisfy the three criteria above, but this is because all three criteria consider low mass particles (criterion 3 does so in at least one side of the symmetry equation), and the assumptions of heavy gas theory are not expected to hold. However, Eq. (2) does agree with the limit of Eq. (3) for high mass.

Leslie (*4*) has recently suggested a similar expression

$$\sigma_T = \frac{2}{A}\frac{\sigma_{fa}}{(1+1/A)^2}\lambda_1 \tag{8}$$

where λ_1 is the first eigenvalue of the free gas scattering operator, and is smaller than the corresponding factor $\sqrt{1 + T/(AT_0)}$ in Eq. (3) by 12% in the case of $A = 1$. The form (8) satisfies criteria 1 and 2 but not criterion 3.

The solution of the diffusion equations for the two overlapping thermal groups in two adjoining half-spaces at temperatures T and T_0 has been given by Selengut (*2*) for the case of constant σ_T. It is repeated here because σ_T assumes different values in the two half-spaces. In Eq. (1) $M_0(v)$ is the number of neutrons per unit velocity interval in the group at temperature T_0 normalized to unity (Eq. (5)), and the flux $\phi_0(x)$ of the group is a function only of space. In the absence of absorption the diffusion equation for $\phi_0(x)$ in the medium at T is

$$\nabla^2\phi_0 - \phi_0/L_0^2 = 0$$

where

$$1/L_0^2 = n\sigma_T(T_0 \to T)/D \tag{9}$$

where D is the diffusion constant. In the medium at T_0 there is a source term from the other group

$$\nabla^2\phi_0 + \phi/L^2 = 0 \tag{10}$$

where $1/L^2 = n\sigma_T(T \to T_0)/D$, assuming the same diffusion constant. (In the medium at T_0, there is of course no net transfer of the flux at T_0 to itself. To be consistent with the fact $\lim_{T\to T_0}\sigma_T \neq 0$, Eq. (10) can be thought of as having equal and opposite source and loss terms which are not shown.)

There are two similar equations in the other group ϕ. The solutions to these four equations, assuming continuity of flux and current at $x = 0$, are

$$\left.\begin{aligned}\phi_0 &= \frac{L_0}{L + L_0}e^{-x/L_0}\\ \phi &= 1 - \frac{L_0}{L + L_0}e^{-x/L_0}\end{aligned}\right\}\text{region } T,\ x > 0$$

$$\left.\begin{aligned}\phi &= \frac{L}{L + L_0}e^{x/L}\\ \phi_0 &= 1 - \frac{L}{L + L_0}e^{x/L}\end{aligned}\right\}\text{region } T_0,\ x < 0$$

An extension can be made to the case of different masses in the two media. Although the expression (3) for σ_T is exact for a free gas, this is not to say that the representation of the flux by two overlapping groups is satisfactory.

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R. M. Pearce

Atomic Energy of Canada Limited
Chalk River, Ontario
Received April 23, 1962

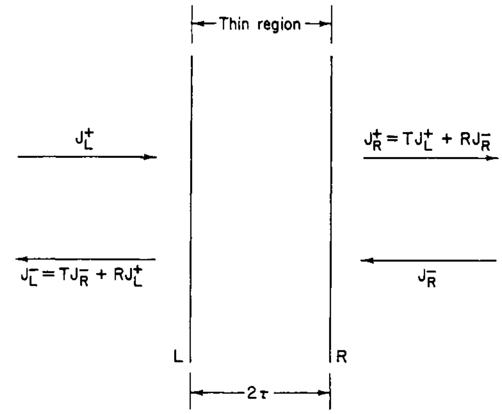

Fig. 1. Surface current relations for a symmetric thin region.

A Comment on the Treatment of Asymmetric Thin Regions*

Thin region theory (*1*) has been used extensively in reactor calculations for the treatment of geometrically thin, but often optically thick, regions. A typical assumption is that the transmitted, reflected, and incident neutron currents near a thin region of thickness $2t$, illustrated in Fig. 1, are approximately isotropic, and that the probabilities of transmission, T, and reflection, R, of neutrons incident on the region are independent of the direction (i.e., right or left) of incidence. The net currents on the left and right faces of the slab are then given by Eq. (7) of ref. *1* as[1]

$$\begin{aligned} J_L &= \frac{1}{2}\frac{1 + T^2 - R^2}{(1+R)^2 - T^2}\varphi_L - \frac{T}{(1+R)^2 - T^2}\varphi_R \\ J_R &= \frac{T}{(1+R)^2 - T^2}\varphi_L - \frac{1}{2}\frac{1 + T^2 - R^2}{(1+R)^2 - T^2}\varphi_R \end{aligned} \tag{1}$$

where φ_L and φ_R are the fluxes at the left and right faces. While Eqs. (1) may be used directly as boundary conditions, a more frequently used form of the theory is based on the fact that the solution of the difference equations form of the diffusion equations for a region without interior mesh points yields expressions for the currents which are of the same form as Eq. (1); viz. (see Eq. (10) of ref. *1*)[1]

$$\begin{aligned} J_L &= \left(\frac{D}{2t} + \Sigma_a t\right)\varphi_L - \frac{D}{2t}\varphi_R \\ J_R &= \frac{D}{2t}\varphi_L - \left(\frac{D}{2t} + \Sigma_a t\right)\varphi_R \end{aligned} \tag{2}$$

By equating coefficients of (1) and (2), a fictitious diffusion coefficient D and a fictitious absorption cross section Σ_a are obtained which allow the *formal* use of diffusion theory within the region, but which in fact preserve the transport theory relations (1) at the surfaces. Modifications of the procedure for the case in which interior mesh points are introduced in the region have also been developed (*2*, *3*).

Thin regions, however, are often introduced into inherently asymmetric configurations. A thin absorbing plate might, for example, be placed between a core and a reflector region in order to suppress a power peak. Then, because of the different spectral indices (*4*) of the materials to the left and right of the thin region, the energy averaged transmission and reflection probabilities for left incidence, T_L and R_L, might be different than the corresponding probabilities, T_R and R_R, for right incidence.[2] In such a case, it is easily shown that Eq. (1) should be replaced by

$$\begin{aligned} J_L &= \frac{1}{2}\frac{T_L T_R + (1 - R_L)(1 + R_R)}{(1 + R_L)(1 + R_R) - T_L T_R}\varphi_L - \frac{T_R}{(1 + R_L)(1 + R_R) - T_L T_R}\varphi_R \\ J_R &= \frac{T_L}{(1 + R_L)(1 + R_R) - T_L T_R}\varphi_L - \frac{1}{2}\frac{T_L T_R + (1 + R_L)(1 - R_R)}{(1 + R_L)(1 + R_R) - T_L T_R}\varphi_R \end{aligned} \tag{3}$$

equations which contain four distinct coefficients, instead of the two which appear in (1).

It is tempting in this case to divide the slab into two regions with fictitious absorption and diffusion constants (Fig. 2), in the expectation that by comparing coefficients of the resulting integrated difference-diffusion equations with (3), the fictitious constants may be determined. However, the relations obtained by this strategem[3] are

$$\begin{aligned} J_L &= \left(\frac{D_1}{t} + \frac{\Sigma_1 t}{2} - \frac{2D_1^2/t}{E}\right)\varphi_L - \frac{2D_1 D_2/t}{E}\varphi_R \\ J_R &= \frac{2D_1 D_2/t}{E}\varphi_L - \left(\frac{D_2}{t} + \frac{\Sigma_2 t}{2} - \frac{2D_2^2/t}{E}\right)\varphi_R \\ E &= 2(D_1 + D_2) + (\Sigma_1 + \Sigma_2)t^2 \end{aligned} \tag{4}$$

which contain, not the necessary four, but only three dis-

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] It is assumed here that the thin region is source free.

[2] If the partial current spectra on each side of the slab were the same, then the left and right transmissions would be the same. There is at present, however, no reason to believe the partial current spectra are not different.

[3] Equations (4) are obtained by integrating the difference-diffusion equation in the slab from the left face to