Letters to the Editor

Comments on Particle Transport in Finite Slabs

In a recent contribution to this Journal, Woolf et al.[1] have derived, through the method of invariant imbedding, a balance equation for the number of interactions suffered by particles in a one-dimensional rod of finite length. In particular, they calculate $T_n(t)$, the transmitted particle current emerging from the right end of the rod of length t after n interactions and $B_n(t)$, the reflected particle current emerging from the left end of the rod after n interactions. The balance equation derived for these quantities is very similar to that which would arise if the method of regeneration points had been used.[2] Indeed, there is a close correspondence between the two methods. The main point of this Letter, however, is not to discuss the many ramifications and interrelationships of invariant imbedding theory, but simply to point out a very concise formulation of the basic equations obtained by Woolf et al. Thus, if we turn to Eqs. (10) through (14b) of their paper, and introduce the following generating functions,

$$G(z;t) = \sum_{n=0}^{\infty} z^n T_n(t) \quad (1)$$

$$H(z;t) = \sum_{n=0}^{\infty} z^n B_n(t) \ , \quad (2)$$

we can multiply Eqs. (10) through (14b) by z^n and sum over n. The result can be written as

$$\lambda \frac{dH(z;t)}{dt} = bzG^2(z;t) \quad (3)$$

$$\lambda \frac{dG(z;t)}{dt} = (fz - 1)\,G(z;t) + bzH(z;t)\,G(z;t) \ , \quad (4)$$

subject to

$$G(z,0) = 1 \quad (5)$$

$$G(0,t) = \exp(-t/\lambda) \quad (6)$$

$$H(z,0) = 0 \quad (7)$$

$$H(0,t) = 0 \ . \quad (8)$$

These equations for the generating functions are easily solved analytically, and after some manipulation we obtain

$$G(z;t) = \frac{(A^2 - B^2)^{1/2}}{(A^2 - B^2)^{1/2}\cosh\left[\frac{t}{\lambda}(A^2 - B^2)^{1/2}\right] - A\sinh\left[\frac{t}{\lambda}(A^2 - B^2)^{1/2}\right]} \quad (9)$$

$$H(z;t) = \frac{B\sinh\left[\frac{t}{\lambda}(A^2 - B^2)^{1/2}\right]}{(A^2 - B^2)^{1/2}\cosh\left[\frac{t}{\lambda}(A^2 - B^2)^{1/2}\right] - A\sinh\left[\frac{t}{\lambda}(A^2 - B^2)^{1/2}\right]} \ , \quad (10)$$

where $A = fz - 1$ and $B = bz$.

The results of Eqs. (15) and (16) of Ref. 1 follow by collecting coefficients of z^n. Thus, it is not necessary to solve the equations for T_n and B_n recursively. Certainly, however, it becomes tedious to carry out the expansions to a high order unless a general term can be identified; we have not been able to do this but neither have we tried very hard. An asymptotic analysis for very large n looks promising.

Some other properties of interest are readily written. For example, the total transmitted current is

$$T(t) = \sum_{n=0}^{\infty} T_n(t) = G(1;t) = \frac{[(f-1)^2 - b^2]^{1/2}}{[(f-1)^2 - b^2]^{1/2}\cosh\left\{\frac{t}{\lambda}[(f-1)^2 - b^2]^{1/2}\right\} - (f-1)\sinh\left\{\frac{t}{\lambda}[(f-1)^2 - b^2]^{1/2}\right\}} \ , \quad (11)$$

which reduces to $1/(1 + bt/\lambda)$ for the conservative case where $f + b = 1$. Similarly, the reflected current is

$$B(t) = \sum_{n=0}^{\infty} B_n(t) = H(1;t) = \frac{b\sinh\left\{\frac{t}{\lambda}[(f-1)^2 - b^2]^{1/2}\right\}}{[(f-1)^2 - b^2]^{1/2}\cosh\left\{\frac{t}{\lambda}[(f-1)^2 - b^2]^{1/2}\right\} - (f-1)\sinh\left\{\frac{t}{\lambda}[(f-1)^2 - b^2]^{1/2}\right\}} \ , \quad (12)$$

which for $f + b = 1$ becomes $bt/(\lambda + bt)$.

A further interesting physical quantity arises if we allow the length of the rod to become semi-infinite, i.e., $t \to \infty$. Then, clearly, $G(z;\infty) = 0$, and we conclude that the transmitted flux is zero. On the other hand, the reflected particle flux is given by

$$H(z;\infty) = \frac{bz}{[(fz-1)^2 - b^2z^2]^{1/2} - fz + 1} \ . \quad (13)$$

Expanding to order z^3, we find

$$H(z;\infty) = \frac{b}{2}z + \frac{fb}{2}z^2 + \frac{1}{8}b(b^2 + 4f^2)z^3 + \ldots \ . \quad (14)$$

Thus, the fraction of particles reflected after one interaction is $b/2$, the fraction after two is $fb/2$, and so on.

[1] STANLEY WOOLF, JOHN C. GARTH, and WILLIAM L. FILIPPONE, *Nucl. Sci. Eng.*, **62**, 278 (1977).
[2] L. JANÓSSY, *Proc. Phys. Soc.*, **63A**, 241 (1950).

The mean value and variance of the distribution are readily obtained from

$$\bar{T}(t) = \frac{\partial G(z;t)}{\partial z} \; ; \qquad z = 1$$

$$\overline{T^2}(t) - \bar{T}^2(t) = \frac{\partial^2 G(z;t)}{\partial z^2} + \frac{\partial G(z;t)}{\partial z} - \left[\frac{\partial G(z;t)}{\partial z}\right]^2 \; ; \quad z = 1 \; .$$

Finally, we note that the space- and angle-dependent problem defined by $T_n(t, \mu, \mu_0)$ and $B_n(t, \mu, \mu_0)$ via Eqs. (24) through (27b) of the paper of Woolf et al. can also be cast into generating function form:

$$\frac{\partial}{\partial t} H(z;t,\mu,\mu_0) = \frac{\mu}{\lambda} \int_{-1}^{0} d\mu'' \int_0^1 d\mu' f(\mu' \to \mu'')\, z\, \Phi(z;t,\mu,\mu'') \times \Phi(z;t,\mu',\mu_0) \; , \tag{15}$$

$$\mu \frac{\partial}{\partial t} \Phi(z;t,\mu,\mu_0) + \frac{1}{\lambda} \Phi(z;t,\mu,\mu_0) = \frac{1}{\lambda} \int_0^1 d\mu' f(\mu' \to \mu)\, z\, \Phi(z;t,\mu',\mu_0) + \frac{1}{\lambda} \int_0^1 d\mu' \int_{-1}^{0} d\mu'' f(\mu' \to \mu'')\, z\, \Phi(z;t,\mu',\mu_0) H(z;t,\mu,\mu'') , \tag{16}$$

where $\mu' \Phi(z;t,\mu',\mu_0) = G(z;t,\mu',\mu_0)$, G and H being defined as above. The boundary conditions are

$$G(z;0,\mu,\mu_0) = \delta(\mu - \mu_0) \tag{17}$$

$$G(0;t,\mu,\mu_0) = \exp(-t/\lambda\mu)\, \delta(\mu - \mu_0) \tag{18}$$

$$H(z;0,\mu,\mu_0) = 0 \tag{19}$$

$$H(0;t,\mu,\mu_0) = 0 \; . \tag{20}$$

These equations are very similar to those based on the backward equation for probability balance introduced into reactor theory by Pál[3] and by Bell.[4]

A further use of the generating function technique can be found in Eq. (42) of Ref. 1, where the n'th collision distribution is given by

$$\phi_n(\zeta) = \phi_0(\zeta) + \frac{1}{2} \int_0^t d\zeta' E_1(|\zeta - \zeta'|)\, \phi_{n-1}(\zeta') \; . \tag{21}$$

Introducing

$$G(z;\zeta) = \sum_{n=0}^{\infty} \phi_n(\zeta) z^n \tag{22}$$

leads to

$$G(z;\zeta) = \frac{\phi_0(\zeta)}{1-z} + \frac{1}{2} z \int_0^t d\zeta' E_1(|\zeta - \zeta'|)\, G(z;\zeta') \; . \tag{23}$$

This equation can be solved by one of several analytic methods, and then the coefficients of z^n can be extracted term by term.

These comments are offered in a spirit of participation and in no way detract from the very interesting and valuable numerical work of Woolf et al. It is hoped that by employing the generating function technique and noting its close similarity with other stochastic processes, a better understanding of these matters will emerge. An example of this technique may be found in two forthcoming papers by the author.[5]

M. M. R. Williams

Queen Mary College
Nuclear Engineering Department
Mile End Road
London E1 4NS, England

February 25, 1977

[3] L. PÁL, *Il. Nuovo Cimento Suppl.*, **VII**, 25 (1958).
[4] G. I. BELL, *Nucl. Sci. Eng.*, **21**, 390 (1965).
[5] M. M. R. WILLIAMS, *Physica* (in press).

Reply to "Comments on Particle Transport in Finite Slabs"

In his Letter, Williams[1] shows how the method of generating functions gives an elegant and useful formulation of the orders-of-scattering approach to particle transport. We applaud his comments and believe that indeed this method enables new insights and analytic results to be obtained on this problem.

In response to his Letter, we would like to offer the following comments:

1. Because of the difficulty of determining the coefficients of z^n, the generating function solution to the one-dimensional transport case, Eqs. (9) and (10) of Ref. 1, does not appear to lead to a more efficient means for determining numerical values for orders-of-scattering results. As an analytic solution, it automatically has the advantage, as does the approach of Bellman et al. [Eqs. (6), (7), and (8) of Ref. 2] and of Mingle,[3] that numerical results at a given thickness do not depend on those at smaller thicknesses. It would be interesting if the polynomial-exponential-product form of the orders-of-scattering solutions [e.g., Eqs. (15) and (16) of Woolf et al.[2]] could be utilized to advantage to develop a more efficient algorithm for evaluating the solution at an arbitrarily high order of scattering for a given thickness.

2. Another author, Abu-Shumays,[4] has previously applied the generating function idea to orders-of-scattering invariant imbedding for transport in a slab. He applies the method to the invariant imbedding equations for the reflection function described by Bellman et al.,[5] and by Wing[6] and obtains results for the average number of collisions of reflected particles and its variance.

3. Williams' Eq. (12) for $B(t)$ is also published in the book by Wing[6] and was derived by a Boltzmann-type approach.

4. We have taken Williams' suggestion and have applied it to the problem of obtaining orders-of-scattering solutions of the time-dependent transport equation.[7,8] The generating function technique shows considerable promise as a tool for obtaining insight in this area.

Stanley Woolf
John C. Garth

Arcon Corporation
North Avenue at Route 128
Wakefield, Massachusetts 01880

William L. Filippone

University of Lowell
Lowell, Massachusetts 01853

March 17, 1977

[1] M. M. R. WILLIAMS, *Nucl. Sci. Eng.*, **63**, 357 (1977).
[2] S. WOOLF, J. C. GARTH, and W. L. FILIPPONE, *Nucl. Sci. Eng.*, **62**, 278 (1977).
[3] J. O. MINGLE, *J. Math. Anal. Appl.*, **38**, 53 (1972).
[4] I. K. ABU-SHUMAYS, *J. Math. Anal. Appl.*, **18**, 453 (1967).
[5] R. E. BELLMAN, R. E. KALABA, and M. C. PRESTRUD, *Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness*, American Elsevier, Inc., New York (1963).
[6] G. M. WING, *An Introduction to Transport Theory*, John Wiley and Sons, Inc., New York (1962).
[7] C. SYROS, *Atomkernenergie*, **16**, 273 (1970).
[8] B. D. GANAPOL and L. M. GROSSMAN, *Nucl. Sci. Eng.*, **52**, 454 (1973).