Letters to the Editor

Comments on "Measurements of Anisotropic Neutron Diffusion Coefficients in Square Lattices of Aluminum in Light Water by the Pulsed Neutron Method"

In a recent Note, Kaneko et al.[1] attribute to me[2] the prediction that the discrete-time eigenvalue cannot exceed the limit of $(\nu\Sigma)_{\min}$ of the materials constituting the heterogeneous system.

I must make it clear that in my *Nukleonik* paper no such statement is made. In fact, the paper deals with the diffusion length experiment and predicts that the spatial decay constant in a heterogeneous system is bounded by $\Sigma_{\min}$. The origin of the time-dependent analog of my result is difficult to trace, but certainly it is evident in the work of Grosshög,[3] Dance and Connolly,[4] and Sjöstrand and Grosshög[5] and, therefore, it is to the work of these authors that reference should have been made by Kaneko et al.

Let me emphasize that this Letter is in no way intended as a criticism of the scientific merit of the paper under discussion, but is simply an effort to set the record straight regarding the role of $(\nu\Sigma)_{\min}$ in heterogeneous systems.

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[1]YOSHIHIKO KANEKO, FUJIYOSHI AKINO, YOSHIRO SUZUOKI, KENJI KITADATE, RYOSUKE KUROKAWA, and KINJI KOYAMA, *Nucl. Sci. Eng.*, **55**, 105 (1974).
[2]M. M. R. WILLIAMS, *Nukleonik*, **12**, 129 (1969).
[3]G. GROSSHÖG, *J. Nucl. Energy*, **24**, 101 (1970).
[4]K. D. DANCE and T. J. CONNOLLY, *J. Nucl. Energy*, **25**, 155 (1971).
[5]N. G. SJÖSTRAND and G. GROSSHÖG, *Proc. Fourth Conf. Peaceful Uses At. Energy*, **7**, 53 (1971).

Comments on "On the Randomness of a Neutron's Kinetic Energy as It Slows Down by Elastic Collisions in an Infinite Medium"

In a recent Note, Barnett[1] considered some problems concerning the slowing down of neutrons in an infinite nonabsorbing medium; he employed the language and the methods of probability theory as an alternative to the procedures typical of transport theory. The purpose of this Letter is to show that Barnett's results can be obtained directly from the appropriate form of the transport equation.

It is my belief that all contributions to a cross-fertilization among the fields of nonequilibrium statistical mechanics, probability theory, and neutron transport theory are very useful. However, in the specific case of Ref. 1, it appears that the interesting features inherent in establishing a connection between transport theory and probability theory can be somewhat hindered by the computational intricacies in which the probability theoretical approach seems to be entangled. Finally, it may be worth mentioning that a debate concerning issues related to those considered here has appeared recently, following a previous Note by Barnett.[2-6]

THE NOTATION

Retaining Barnett's notation we let

$A \triangleq$ scattering mass of nucleus in neutron mass units $(A \geq 1)$

$$\alpha \triangleq \left(\frac{A-1}{A+2}\right)^2 , \text{ so that } 0 \leq \alpha < 1$$

$$\beta \triangleq -\log(\alpha)$$

$$H(x) \triangleq \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$I_{(b,c)}(x) \triangleq H(x-b) - H(x-c) , \quad \text{for} \quad b < c$$

$Q_n \triangleq$ kinetic energy of neutron during n'th flight

$Q_{\max} \triangleq$ maximum kinetic energy of injected neutrons

$u_n \triangleq \log(Q_{\max}/Q_n) =$ lethargy during n'th flight

$E[\ldots] \triangleq$ expectation value of the random variable $[\ldots]$

$P[\ldots] \triangleq$ probability that $[\ldots]$ is true

$M(u) \triangleq$ number of collisions at which the neutron's lethargy crosses the value u.

Furthermore, we employ the following notation, which differs from Barnett's:

$p_n(u) \triangleq$ probability density function for the lethargy of neutrons in their n'th flight

$p_n^*(Q) \triangleq$ probability density function for the energy of neutrons in their n'th flight

$q(u) \triangleq$ probability density function for the lethargy of injected neutrons.

[1]C. S. BARNETT, *Nucl. Sci. Eng.*, **55**, 234 (1974).
[2]C. S. BARNETT, *Nucl. Sci. Eng.*, **50**, 398 and **52**, 381 (1973).
[3]G. C. POMRANING, *Nucl. Sci. Eng.*, **52**, 144 (1973).
[4]E. R. COHEN, *Nucl. Sci. Eng.*, **52**, 280 (1973).
[5]E. H. CANFIELD, *Nucl. Sci. Eng.*, **53**, 137 (1974).
[6]B. D. GANAPOL, *Nucl. Sci. Eng.*, **53**, 350 (1974).

Here the requirements

$$\int_0^\infty du p_n(u) = \int_0^{Q_{\max}} dQ\, p_n^*(Q) = \int_0^\infty du\, q(u) = 1$$

must be satisfied.

Finally, we define the Laplace transform pairs

$$\begin{cases} \tilde{g}(s) = \mathcal{L}[g(.)] = \int_0^\infty du \exp(-su)\, g(u) \\ g(u) = \mathcal{L}^{-1}[\tilde{g}(.)] \quad . \end{cases} \tag{1}$$

LETHARGY DISTRIBUTIONS

Under the assumption that all neutrons generated by the applied source have energy $Q \leq Q_{\max}$ and that there is no absorption or leakage, the neutron transport equation can be written in the form[7]

$$p_1(u) = q(u) \tag{2}$$

$$p_{n+1}(u) = \int_0^u du'\, p_n(u') f(u' - u) \quad , \quad n = 1, 2, \ldots \quad , \tag{3}$$

where

$$f(w) = \frac{e^{-w}}{1 - \alpha} I_{(0,\beta)}(w) \quad . \tag{4}$$

Here $\int_0^\infty dw\, f(w) = 1$. Laplace transforming Eqs. (2) and (3) we find

$$\tilde{f}(s) = \mathcal{L}[f(.)] = \frac{1 - \exp[-(1+s)]\,\beta}{(1 - \alpha)(1 + s)} \tag{5}$$

$$\tilde{p}_{n+1}(s) = \tilde{q}(s)\tilde{f}^n(s) \quad , \quad \text{for} \quad n = 0, 1, 2 \ldots \quad . \tag{6}$$

The function $f(w)$, defined by Eq. (4), can be interpreted as the probability density function for the random variable "lethargy gain per collision." Then, if we let

$$\tilde{f}^{(k)} \triangleq \frac{d^k}{ds^k} \tilde{f}(s)\Big|_{s=0} \quad , \tag{7}$$

we find

$$E[w] = \int_0^\infty dw\, f(w) w = -\tilde{f}^{(1)} = 1 + \frac{\alpha}{1 + \alpha} \log(\alpha) \tag{8a}$$

$$E[w^2] = \int_0^\infty dw\, f(w) w^2 = \tilde{f}^{(2)}$$

$$= \frac{1}{1 - \alpha} \{2 - \alpha - \alpha[\log(\alpha) - 1]^2\} \quad . \tag{8b}$$

Results in Eq. (8) were obtained by Barnett by the same procedure [see Eqs. (14) and (15) of Ref. 1]. Similarly, Barnett's results in Eqs. (16) and (17) can be obtained from Eq. (6) of this Letter.

ENERGY AVERAGES

Employing definition (1) and remembering that $Q(u) = Q_{\max} \exp(-u)$, we find for $a > -1$ and $n = 1, 2, \ldots$

$$E[Q_n^a] = \int_0^\infty du\, Q^a(u)\, p_n(u) = \int_0^\infty du\, Q_{\max}^a \exp(-au)\, p_n(u)$$

$$= Q_{\max}^a \tilde{p}_n(a) \quad , \tag{9}$$

so that, making use of Eqs. (5) and (6),

$$(E[Q_n^a])(E[Q_1^a])^{-1} = \tilde{f}^{n-1}(a) = \left[\frac{1 - \alpha^{(1+a)}}{(1 - \alpha)(1 + a)}\right]^{n-1} \quad , \tag{10}$$

which is Barnett's[1] result in Eq. (1).

HYDROGEN SCATTERING ($A = 1$) WITH A MONOENERGETIC SOURCE

Here we have

$$q(u) = p_1(u) = \delta(u) \quad . \tag{11}$$

Also, $A = 1$, so that $\alpha = 0, \beta \to +\infty$, $\tilde{f}(s) = (1 + s)^{-1}$. Then, for $n = 1, 2, \ldots$,

$$\tilde{p}_{n+1}(s) = \tilde{f}^n(s) = (1 + s)^{-n} \tag{12}$$

$$p_{n+1}(u) = \mathcal{L}^{-1}[(1 + s)^{-n}] = e^{-u} \frac{u^{n-1}}{(n-1)!} H(u) \quad . \tag{13}$$

Hence, for $n = 1, 2, 3 \ldots$

$$p_{n+1}^*(Q) = p_{n+1}[u(Q)]\left(\left|\frac{dQ}{du}\right|\right)^{-1} = \frac{[\log(Q_{\max}/Q)]^{n-1}}{(n-1)!\, Q_{\max}}$$

$$\times I_{(0, Q_{\max})}(u) \quad . \tag{14}$$

Results (13) and (14) coincide with Barnett's[1] results, Eqs. (19) and (20). We turn now to the evaluation of $P[M(u) = N]$ for the case of hydrogen under the assumption of validity of Eq. (11). It is clear that

$$P[M(u) = 1] = H(u) \int_u^\infty du'\, f(u') = e^{-u} H(u) \quad . \tag{15}$$

Moreover, for $n = 2, 3, \ldots$

$P[M(u) = n] = P$ (lethargy u_n of neutron during its n'th flight is smaller than u) $- P[u_{n+1} < u]$

$$= \int_0^u dy\, p_n(y) - \int_0^u dy\, p_{n+1}(y)$$

$$= H(u) \int_0^u dy \frac{d}{dy}\left[e^{-y} \frac{y^{n-1}}{(n-1)}\right]$$

$$= e^{-u} \frac{u^{n-1}}{(n-1)!} H(u) \quad , \tag{16}$$

where Eq. (13) was employed. It is easy to verify that for all $u > 0$, $\sum_{n=0}^{\infty} P[M(u) = n] = 1$, as expected. Equations (15) and (16) coincide with Barnett's Eq. (22).

SCATTERING WITH $A \geq 2$ AND MONOENERGETIC SOURCE

Assumption (11) is retained. Employing the procedure yielding Eqs. (15) and (16) and making use of the definition in Eq. (4), we find

$$P[M(u) = 1] = H(u) \int_u^{+\infty} dy\, f(y) = \frac{1}{1 - \alpha} (e^{-u} - \alpha) I_{(0,\beta)}(u) \quad . \tag{17}$$

Moreover, making use of Eq. (6),

$$P[M(u) = n] = \int_0^u dy\, p_n(y) - \int_0^u dy\, p_{n+1}(y)$$

$$= \mathcal{L}^{-1}[s^{-1}\tilde{f}^{n-1}(s)] - \mathcal{L}^{-1}[s^{-1}\tilde{f}^n(s)] \quad n = 2, 3, \ldots \quad . \tag{18}$$

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[7] J. FERZIGER and P. F. ZWEIFEL, *The Theory of Neutron Slowing Down in Nuclear Reactors*, The MIT Press, Cambridge, Massachusetts (1966).

Now let[8]

$$m_n(u) \triangleq \left[e^u - \sum_{R=0}^{n-1} \frac{u^R}{R!}\right] H(u) = \mathcal{L}^{-1}\left[(s-1)^{-1} s^{-n}\right] , \quad n = 1, 2, \ldots . \tag{19}$$

Then, making use of the definition in Eq. (4), one finds[8]

$$\begin{aligned}\mathcal{L}^{-1}\left[s^{-1}\bar{f}^n(s)\right] &= (1-\alpha)^{-n} e^{-u} \mathcal{L}^{-1}\{(s-1)^{-1} s^{-n} \times [1 - \exp(-s\beta)]^n\} \\ &= (1-\alpha)^{-n} e^{-u} \sum_{k=0}^{n} \mathcal{L}^{-1}\left[\binom{n}{k}(-1)^k \exp(-s\beta k)(s-1)^{-1} s^{-n}\right] \\ &= (1-\alpha)^{-n} e^{-u} \sum_{k=0}^{n} (-1)^k \binom{n}{k} H(u-\beta k)\, m_n(u-\beta k)\end{aligned} \tag{20}$$

for $n = 1, 2, \ldots$. Substitution of Eq. (20) into Eq. (18) yields the value of $P[M(u) = n]$. The result coincides with Barnett's[1] Eq. (53).

Making use of Eq. (18) we find

$$\begin{aligned}E[M(u)] &\triangleq \sum_{n=1}^{\infty} n P[M(u) = n] \\ &= \mathcal{L}^{-1}\left\{\sum_{n=1}^{\infty} n s^{-1}\left[\bar{f}^{n-1}(s) - \bar{f}^n(s)\right]\right\} \\ &= \mathcal{L}^{-1}\{s^{-1}[1 - \bar{f}(s)]^{-1}\}\end{aligned} \tag{21a}$$

$$\begin{aligned}E[M^2(u)] &\triangleq \sum_{n=1}^{\infty} n^2 P[M(u) = n] \\ &= \mathcal{L}^{-1}\left\{\sum_{n=1}^{\infty} n^2 s^{-1}\left[\bar{f}^{n-1}(s) - \bar{f}^n(s)\right]\right\} \\ &= \mathcal{L}^{-1}\left[s^{-1}\sum_{n=0}^{\infty}(2n+1)\bar{f}^n(s)\right] \\ &= \mathcal{L}^{-1}\{s^{-1}[1 + \bar{f}(s)][1 - \bar{f}(s)]^{-2}\} .\end{aligned} \tag{21b}$$

Equation (21) shows that $E[M(u)]$ is the integral with respect to u of the function $\chi(u)$, which is discussed in Sec. II of Ref. 7. Now we observe that $[1 - \bar{f}(0)] = 0$. Furthermore, if we let $x = Re(s)$ and $\omega = Im(s)$, we find

$$|\bar{f}(x + i\omega)| \leq \int_0^{\infty} du \exp(-xu) f(u) < 1 , \quad \text{for} \quad x > 0$$

$$|Re\, \bar{f}(i\omega)| = \left|\int_0^{\infty} du \cos(\omega u) f(u)\right| < 1 , \quad \text{for} \quad \omega \neq 0 .$$

Hence, in the complex plane, the origin is the rightmost singularity[9] of $[1 - \bar{f}(s)]^{-1}$. Now, known theorems[10] link the behavior of the transformed function $\bar{g}(s)$ in the neighborhood of its rightmost singularity with the behavior of $g(u) = \mathcal{L}^{-1}[\bar{g}(.)]$ for $u \to +\infty$. Employing definition (7) we find

$$s^{-1}[1 - \bar{f}(s)]^{-1} = s^{-2} \cdot \frac{-1}{f^{(1)}} + \frac{s^{-1}}{2} \frac{f^{(2)}}{[f^{(1)}]^2} + O(1) , \quad \text{for } s \to 0 \tag{22a}$$

$$\begin{aligned}s^{-1}[1 + f(s)][1 - f(s)]^{-2} &= \frac{1}{[f^{(1)}]^2}\left(2s^{-3} + s^{-2}\left[f^{(1)} - 2\frac{f^{(2)}}{f^{(1)}}\right] + s^{-1} \times \left\{\frac{3}{2}\left[\frac{f^{(2)}}{f^{(1)}}\right]^2 - \frac{2}{3}\frac{f^{(3)}}{f^{(1)}} - \frac{f^{(2)}}{2}\right\}\right) + O(1) , \quad \text{for } s \to 0 .\end{aligned} \tag{22b}$$

Then[10] we are entitled to claim that

$$E[M(u)] = \frac{u}{E[w]} + \frac{1}{2}\frac{E[w^2]}{(E[w])^2} + o(u^{-1}) , \quad \text{for} \quad u \to +\infty \tag{23a}$$

$$\begin{aligned}E[M^2(u)] &= \left(\frac{u}{E[w]}\right)^2 + \frac{u}{E[w]}\left\{2\frac{E[w^2]}{(E[w])^2} - 1\right\} \\ &+ \left\{\frac{3}{2}\frac{(E[w^2])^2}{(E[w])^4} - \frac{2}{3}\frac{E[w^3]}{(E[w])^3} - \frac{1}{2}\frac{E[w^2]}{(E[w])^2}\right\} + o(u^{-1}) , \quad \text{for} \quad u \to +\infty ,\end{aligned} \tag{23b}$$

where Eqs. (8) were employed. Result (23a) yields Barnett's[1] Eq. (43). Moreover, by combining Eqs. (23a) and (23b) and after some manipulations, Barnett's[1] result (54) can be obtained.

SOME CONCLUSIONS

Classical transport theory has been employed to derive results that Barnett[1] obtained by a probability-theoretical method; thus, the scope of classical transport theory has been shown not to be limited to the evaluation of average behaviors.

The transport theory approach seems to involve simpler mathematical procedures. However, it must be recognized that results given in Eq. (23) have been obtained here by a mathematical procedure that is only heuristic. A rigorous treatment would require proof that the series and the inverse transformation appearing in Eqs. (21a) and (21b) commute. Furthermore, the transition from Eqs. (22a) and (22b)—where the O symbol appears—to Eqs. (23a) and (23b)—where the o symbol appears—ought to be justified by checking to see that all assumptions of the relevant theorems[10] are satisfied.

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[8]F. OBERHETTINGER and L. BADII, *Tables of Laplace Transforms*, Springer-Verlag, New York (1973).

[9]The location of the real roots of $[1 - \bar{f}(s)] = 0$ has been studied recently by C. E. SIEWERT and A. R. BURKARDT, *Nucl. Sci. Eng.*, **54**, 455 and **55**, 247 (1974).

[10]G. DOETSCH, *Handbuch der Laplace Transformation*, Vols. I and II, Verlag Birkhauser, Basel (1950 and 1956).

Response to "Comments on 'On the Randomness of a Neutron's Kinetic Energy as It Slows Down by Elastic Collisions in an Infinite Medium' "

It is, of course, rewarding to see the results confirmed by a different method, particularly when that method is more familiar to readers of *Nuclear Science and Engineer-*