Letter to the Editors

On the Energy-Dependent Transport Equation*

Ferziger and Leonard (*1*) have solved the energy-dependent neutron transport equation, using the techniques developed by Case (*2*) for the one-speed problem. Ferziger and Leonard make two basic assumptions to obtain their solution–isotropic scattering and energy-independent cross sections. They then obtain some explicit answers for the case that the moderator is a heavy gas.

Zelazny (*3*) has studied a similar problem, however, *without* invoking the approximation of isotropic scattering, although his method of attack is considerably less amendable to practical application than is the method of ref. *1*.

A casual inspection of the problem would seem to indicate that the assumption of isotropic scattering is inconsistent with a model in which the neutron energy changes in an elastic collision, since both the scattering anisotropy and the energy change are of order $1/A$ (A = ratio of moderator to neutron mass). However, the purpose of this note is to show that within the context of the heavy gas model (*4*), the anisotropic scattering contribution is actually of order σ_a/A, and so may be considered to be of order $(1/A)^2$ (*4*). Thus, the procedure of Ferziger and Leonard is justified, and the additional complications introduced by Zelazny's treatment are not necessary.

On the other hand, the heavy gas model explicitly predicts an energy-dependent cross section, the energy-dependent term being inversely proportional to the neutron energy. Thus the assumption of constant cross section, even if the variation of absorption cross section with energy may be neglected, is open to question, unless the results are applied at moderately high energies, so that the T/AE term may be ignored. This may not be entirely unreasonable, since the heavy gas model requires that

$$T/AE \ll 1$$

The transport equation for plane symmetry, and in the absence of sources is

$$\mu \frac{\partial \psi}{\partial x}(x, E, \mu) + [\sigma_a(E) + \sigma_s(E)]\psi(x, E, \mu) = \iint dE'\, d\Omega' \sigma(E' \rightarrow E, \mathbf{\Omega}' \cdot \mathbf{\Omega})\psi(x, E' \mu') \tag{1}$$

where

$$\sigma_s(E) = \iint \sigma(E \rightarrow E', \mathbf{\Omega}' \cdot \mathbf{\Omega})\, dE'\, d\Omega' \tag{2}$$

* Work supported by the U. S. Atomic Energy Commission.

Applying detailed balance and translational invariance to Eq. (1), we find

$$[\sigma_a(E) + \sigma_s(E) - \mu/\nu]\chi(\nu, E, \mu) = \iint dE'\, d\Omega' \sigma(E \rightarrow E', \mathbf{\Omega}' \cdot \mathbf{\Omega})\chi(\nu, E', \mu'), \tag{3}$$

where

$$e^{-x/\nu}\chi(\nu, E, \mu) = \psi(X, E, \mu)/M(E)$$

and $M(E)$ is the maxwellian distribution.

In the heavy gas approximation, $\sigma(E \rightarrow E', \mathbf{\Omega}' \cdot \mathbf{\Omega})$ is (*4*)

$$\sigma(E \rightarrow E', \mathbf{\Omega}' \cdot \mathbf{\Omega}) = \frac{\sigma_0}{4\pi}\sqrt{E'/E}\left\{\delta(E' - E) + \frac{1}{A}[E' + E - 2\mathbf{\Omega}' \cdot \mathbf{\Omega}\sqrt{E'E}][\delta'(E' - E) + T\delta''(E' - E)]\right\} + O(1/A^2), \tag{4}$$

where σ_0 is the bound scattering cross section of the moderator.

Using Eq. (4) we can carry out the integrations over dE' and $d\Omega'$ in Eqs. (2) and (3) to obtain:

$$\left[\sigma_a(E) + \sigma_s(E) - \frac{\mu}{\nu}\right]\chi(\nu, E, \mu) = \left[\frac{\sigma_s(E)}{2} + \frac{\sigma_0\, \Theta_E}{A}\right]\chi_0(\nu, E) + \left[\frac{\sigma_0}{A} - \frac{\sigma_0\, \Theta_E}{A}\right]\mu\chi_1(\nu, E) \tag{5}$$

where

$$\chi_l(\nu, E) = \int_{-1}^{1} d\mu P_l(\mu)\chi(\nu, E, \mu)$$

and

$$\Theta_E = (2T - E)\frac{\partial}{\partial E} + ET\frac{\partial^2}{\partial E^2}$$

and

$$\sigma_s(E) = \sigma_0\left[1 - \frac{1}{A}\left(2 - \frac{T}{2E}\right)\right] \tag{6}$$

If we multiply Eq. (5) by $d\mu$ and integrate, we see that $\chi_1(\nu, E)$ is proportional to $\chi_0(\nu, E)$

$$\chi_1(\nu, E) = \nu\left[\sigma_a(E) - \frac{2\sigma_0\, \Theta_E}{A}\right]\chi_0(\nu, E) \tag{7}$$

Since $\sigma_a(E)$ should be considered to be proportional to $1/A$ (*4*), we can see from Eq. (7) that $\chi_1(\nu, E)$ is proportional to $1/A$. To be consistent with the heavy gas approximation, we neglect terms of order $1/A^2$ in Eq. (5). This gives

$$\left[\sigma_a(E) + \sigma_s(E) - \frac{\mu}{\nu}\right]\chi(\nu, E, \mu) = \left[\frac{\sigma_s(E)}{2} + \frac{\sigma_0\,\Theta_E}{A}\right]\chi_0(\nu, E) \tag{8}$$

Except for the energy dependence of $\sigma_s(E)$, this is just the expression used by Ferziger and Leonard. Thus we see that in the heavy gas approximation the scattering kernal is isotropic. However, $\sigma_s(E)$ is given by Eq. (6) and the scattering cross section is energy dependent.

These results have been derived for an ideal gas. However, Aamodt *et al.* (*5*) have shown that the classical limit of $\sigma(E' \to E, \mathbf{\Omega}' \cdot \mathbf{\Omega})$ for any state of the moderator is the ideal gas result. Thus the arguments given in this letter are valid for any heavy moderator to the extent that the moderator can be approximated as a classical system.

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Received November 5, 1962