$$\frac{dA_{l0}(\rho)}{d\rho} = \int_1^\infty f'(t\rho)\frac{dt}{t^{l+1}} = \frac{1}{\rho}\int_1^\infty \frac{1}{t^{l+1}}\,df(t\rho)$$

$$= \frac{1}{\rho}\left[\frac{f(t\rho)}{t^{l+1}}\right]_1^\infty + \frac{l+1}{\rho}\int_1^\infty f(t\rho)\frac{dt}{t^{l+2}} \qquad (4)$$

$$= \frac{1}{\rho}\left[(l+1)A_{l0}(\rho) - f(\rho)\right].$$

Once $f(\rho)$ has been obtained, these simple first order differential equations can be solved numerically *for each l*. Bessel's equation can be used to derive a differential equation that provides a rapid numerical method for determining $f(\rho)$:

$$f''(\rho) + \left(2 + 2\alpha + \frac{1}{\rho}\right)f'(\rho) + \left[\alpha(2+\alpha) + \frac{(1+\alpha)}{\rho}\right]f(\rho) = 0. \qquad (5)$$

For higher values of n, $A_{ln}(\rho)$ can be obtained by differentiating Eq. (2):

$$-2A_{l,n+1}(\rho) = \alpha A_{l,n}(\rho) + A'_{l+1,n}(\rho), \qquad (6)$$

where, as in Eq. (5), the prime means a derivative with respect to ρ. Equation (6) can be used in conjunction with Eq. (4). For a given accuracy, the use of successively higher derivatives would, of course, severely limit the coarseness of the mesh size used, but since $A_{ln}(\rho)$ vanishes very rapidly with ρ as n increases, high accuracy here is not really required.

All that remains to be done is to find starting and possible ending solutions for Eqs. (4) and (5). We note first of all that the homogeneous solution to Eq. (4) is ρ^{l+1}, which becomes infinite for large ρ and, therefore, must have a zero coefficient. For small ρ, multiplying the power series expansions for $e^{-(1+\alpha)\rho}$ and $I_0(\rho)$ together gives

$$f(\rho) = 1 - (1+\alpha)\rho + [\tfrac{1}{2}(1+\alpha)^2 + \tfrac{1}{4}]\rho^2 \cdots. \qquad (7)$$

The power series expansion for $A_{l0}(\rho)$ can be obtained by inserting Eq. (7) into Eq. (4) and equating the coefficients of equal powers of ρ:

$$A_{l0}(\rho) = B_{l+1}(\rho) - (1+\alpha)B_l(\rho)\rho + [\tfrac{1}{2}(1+\alpha)^2 + \tfrac{1}{4}]B_{l-1}(\rho)\rho^2 \cdots, \qquad (8)$$

where the functions $B_l(\rho)$ are defined:

$$B_l(\rho) = 1/l \quad \text{if} \quad l \neq 0$$
$$= -ln\rho \quad \text{if} \quad l = 0.$$

For large ρ, the properties of $I_0(\rho)$ tell us that

$$f(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{-\alpha\rho}\left[1 + \frac{1}{8\rho} + \frac{9}{128\rho^2} + \cdots\right]. \qquad (9)$$

By writing

$$A_{l0}(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{-\alpha\rho}\left[a + \frac{b}{\rho} + \frac{c}{\rho^2} + \cdots\right], \qquad (10)$$

inserting the expression in Eq. (4), and equating coefficients of equal powers of ρ, we obtain:

$$a = 0, b = 1/\alpha, c = 1/(8\alpha) - (l + \tfrac{5}{2})/\alpha^2 \quad \text{if} \quad \alpha\rho \gg 1,$$

$$a = 1/(l + \tfrac{3}{2}),\ b = (\tfrac{1}{8})/(l + \tfrac{5}{2}),\ c = (\tfrac{9}{128})/(l + \tfrac{7}{2}) \quad \text{if} \quad \alpha = 0. \qquad (11)$$

Finally, it should be mentioned that if $f(\xi)$ is generalized to a form that allows $\psi(x)$ to be the Doppler broadened resonance function, Eqs. (3, 4, and 6) will still be valid.

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HARVEY J. AMSTER

Department of Nuclear Engineering
University of California
Berkeley, California

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Treatment of Annular Voids in Diffusion Theory

The usual one-dimensional multigroup diffusion codes can be used to calculate configurations with annular void regions by assigning a fictitious diffusion coefficient to the void region. The recipe is suitable for both infinite cylinders and spheres. Basically, the method is to calculate a diffusion coefficient for the void region which preserves the annular void boundary conditions consistent with the neutron streaming problem in the P-1 approximation.

In the cylindrical case, let us assume that the void region, $r_1 \leqq r \leqq r_2$, is characterized by some value of $\kappa^2 = \Sigma_a/D$ where $\Sigma_a \ll D$. Diffusion theory gives

$$\phi_v(r) = AI_0(\kappa r) + BK_0(\kappa r) \qquad r_1 \leqq r \leqq r_2 \quad (1)$$

where $\phi_v(r)$ is the void neutron flux and I_0 and K_0 are the modified Bessel functions. For $\kappa r \ll 1$, Eq. (1) becomes

$$\phi_v(r) = A + B\ln\kappa r. \qquad (2)$$

By differentiation, B can be expressed as $r(\partial\phi_v/\partial r)$. Using the continuity of the neutron current across the interfaces at r_1 and r_2, we have

$$B = \frac{D_1 r_1}{D}\left(\frac{\partial\phi}{\partial r}\right)_1 = \frac{D_2 r_2}{D}\left(\frac{\partial\phi}{\partial r}\right)_2 \qquad (3)$$

where $D_1(D_2)$ represents the diffusion coefficient for the region $r < r_1 (r > r_2)$, D is the void "diffusion coefficient" and the derivatives are evaluated for the diffusing regions at the void interfaces. Using Eqs. (2) and (3) and assuming continuity of the neutron flux at the interfaces gives

$$\phi(r_1) - \phi(r_2) = -\frac{D_1 r_1}{D}\left(\frac{\partial\phi}{\partial r}\right)_1 \ln\left(\frac{r_2}{r_1}\right). \qquad (4)$$

The P-3 approximation to the neutron streaming problem across an annular void in cylindrical geometry has been derived by Tait (*1*). Using only the P-1 terms and assuming the diffusion approximation, i.e., that the net neutron current is given by $-D(\partial\phi/\partial r)$, Tait's equations give

$$r_1 D_1 \left(\frac{\partial \phi}{\partial r}\right)_1 = r_2 D_2 \left(\frac{\partial \phi}{\partial r}\right)_2 \tag{5}$$

$$\frac{1}{2}\phi(r_1) + D_1\left(\frac{\partial\phi}{\partial r}\right)_1 = \frac{1}{2}\phi(r_2) + \left(\frac{2}{\pi}\right) D_2 \left(\frac{\partial\phi}{\partial r}\right)_2 \left\{\frac{r_2}{r_1}\sin^{-1}\left(\frac{r_1}{r_2}\right) + \left[1 - \left(\frac{r_1}{r_2}\right)^2\right]^{1/2}\right\}. \tag{6}$$

Note that diffusion theory and P-1 give the same condition on the current; cf. Eqs. (5) and (3). Using Eq. (5), Eq. (6) becomes

$$\phi(r_1) - \phi(r_2) = -2D_1\left(\frac{\partial\phi}{\partial r}\right)_1 \cdot \left\{1 - \frac{2}{\pi}\left[\sin^{-1}\left(\frac{r_1}{r_2}\right) + \frac{r_1}{r_2}\sqrt{1 - \left(\frac{r_1}{r_2}\right)^2}\right]\right\}. \tag{7}$$

Equating Eqs. (7) and (4) and solving the resulting expression for the void diffusion coefficient gives

$$D/r_1 = \frac{\ln(r_2/r_1)}{2\{1 - (2/\pi)\,[\sin^{-1}(r_1/r_2) + (r_1/r_2)\sqrt{1 - (r_1/r_2)^2}]\}}. \tag{8}$$

Therefore, by using the above expression for the void diffusion coefficient in diffusion theory calculations the P-1 boundary conditions for an annular void region are preserved. Note that D is purely a function of geometry and hence group independent. For $(r_1/r_2) \ll 1$, Eq. (8) becomes

$$D/r_1 = \frac{\ln(r_2/r_1)}{2[1 - (4/\pi)(r_1/r_2)]} \tag{9}$$

and hence $(D/r_1) = \frac{1}{2}\ln(r_2/r_1)$ as $(r_1/r_2) \to 0$. Table I gives values of (D/r_1) as a function of the ratio (r_1/r_2). It should be noted that the above treatment is also valid for $\Sigma_a = 0$.

An analogous expression for the void diffusion coefficient for a spherical annular region can similarly be derived. Extending the material as given by Davison (*2*), the annular void boundary conditions for the spherical case are

$$r_1^2\left(\frac{\partial\phi}{\partial r}\right)_1 = r_2^2\left(\frac{\partial\phi}{\partial r}\right)_2 \tag{10}$$

$$\frac{1}{2}\phi(r_1) + D_1(\partial\phi/\partial r)_1 = \frac{1}{2}\phi(r_2) + D_2\left(\frac{\partial\phi}{\partial r}\right)_2\left\{1 - \left[1 - \left(\frac{r_1}{r_2}\right)^2\right]^{3/2}\right\}. \tag{11}$$

Following the same procedure as in the cylindrical case gives

$$\frac{D}{r_1} = \frac{1}{2}\left[1 - \frac{r_1}{r_2}\right]\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]^{-3/2} \tag{12}$$

for the void diffusion coefficient. Again the above value of D in a diffusion theory calculation preserves the conditions expressed by Eqs. (10) and (11).

As is well known, a void region in plane geometry can simply be neglected since the void does not contribute to the optical thickness. However, it is interesting to consider the plane case in light of the above formulation. For the plane case, the equation corresponding to Eq. (4) is given by

TABLE I

VALUES OF D/r_1

r_1/r_2	D/r_1	
	Cylinder	Sphere
0.99	4.2427	1.7811
0.9	1.4089	0.60373
0.8	1.0719	0.46296
0.7	0.94799	0.41185
0.6	0.89696	0.39063
0.5	0.88637	0.38490
0.4	0.90788	0.38967

$$\phi(x_1) - \phi(x_2) = \frac{D_1}{D}\left(\frac{\partial\phi}{\partial x}\right)_1 (x_1 - x_2). \tag{13}$$

To find the value of D for the plane case, we make the substitution $r_2 = (r_1 + t)$ in Eq. (12) for $t \ll r_1$, r_2 and consider $r_1 \to \infty$; this gives

$$D = (r_1^3/2^5 t)^{1/2}, \tag{14}$$

hardly a surprising result since as $r_1 \to \infty$ the spherical case approaches the plane case with a void region of thickness t and hence the right hand side of Eq. (13) vanishes giving the expected result that $\phi(x_1) = \phi(x_2)$.

It should be emphasized that for the cylindrical and slab cases, the treatment is restricted to infinite cylinders and infinite planes; i.e., no end leakage. The end leakage is a separate problem.

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E. GARELIS

Vallecitos Atomic Laboratory
General Electric Company
Pleasanton, California
Received November 27, 1961

On the Influence of Pressure on Boiling Water Reactor Dynamic Behavior at Atmospheric Pressure

Due to the present importance of understanding reactor kinetics, it is essential to be as rigorous as possible in the mathematical modeling of reactor problems. Regrettably, such has not been the case in most analyses in this field. On the other hand very great care has been taken with the analysis of the hydrodynamic fields. Overwhelming mathematical detail has been introduced, with the result that integration can be done only on a computer. However, the basic equation of reactor kinetics has been modeled incorrectly.