Eq. (7) becomes:

$$\delta' = -\frac{1}{2}\iint dE dE' \varphi(E)\varphi(E') \cdot \frac{(\Sigma_2 \Sigma_1' - \Sigma_1 \Sigma_2')^2}{\Sigma_1 \Sigma_2 \Sigma_1' \Sigma_2' (\Sigma_1 + \Sigma_2)(\Sigma_1' + \Sigma_2')} \leqq 0 \tag{8}$$

This completes the proof of (3).

Another useful result is that a limit can be placed upon the difference between the extremes of (3). Defining

$$\epsilon = \frac{\Sigma^{\mathrm{b}}_{\mathrm{tr}\,i} - \Sigma^{\mathrm{a}}_{\mathrm{tr}\,i}}{\Sigma^{\mathrm{a}}_{\mathrm{tr}\,i}} = \frac{\sum_{j=1}^{N} \{\langle \Sigma_j \rangle - \langle 1/\Sigma_j \rangle^{-1}\}}{\sum_{j=1}^{N} \langle 1/\Sigma_j \rangle^{-1}}$$

we see that ϵ can be written in the form:

$$\epsilon = \frac{\sum_{j=1}^{N} \langle 1/\Sigma_j \rangle^{-1} \{\langle \sigma_j \rangle \langle 1/\sigma_j \rangle - 1\}}{\sum_{j=1}^{N} \langle 1/\Sigma_j \rangle^{-1}} \tag{9}$$

from which it follows that:

$$[\langle \sigma_j \rangle \langle 1/\sigma_j \rangle - 1]_{\min} \leqq \epsilon \leqq [\langle \sigma_j \rangle \langle 1/\sigma_j \rangle - 1]_{\max} \tag{10}$$

where $[\cdots]_{\min}$ means that value of the bracketed quantity for that element in the mixture which makes the quantity a minimum and $[\cdots]_{\max}$ has an analogous definition.

An interesting application of these results involves the definition of δD, the perturbation in the diffusion coefficient to be used in the calculation of a reactivity coefficient. We consider the unperturbed system to have a transport cross section $\Sigma^0_{\mathrm{tr}} = a/E^{1/2}$, where a is a constant, and introduce a constant perturbation $\Sigma'_{\mathrm{tr}} = 0.1\ a$. Calculating over an energy group, between limits of 1 and 2 Mev, and assuming for convenience a flux spectrum proportional to E^{-1}, the following results are obtained:

$$\delta D = -0.043\ a^{-1}$$

$$\delta D_{\mathrm{a}} = -0.042\ a^{-1}$$

$$\delta D_{\mathrm{b}} = -0.041\ a^{-1}$$

The quantity δD was calculated exactly while δD_{a} and δD_{b} were obtained from (2a) and (2b) respectively. The expression for δD, given in a report (unpublished) describing the AIM-6 Multigroup Diffusion Equation Code, $\delta D = -D_0^2 \delta(1/D)$ gives a result of 0.038 a^{-1}, which is about 12% too low.

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P. F. Zweifel*
G. L. Ball

Atomic Power Development Associates Inc.
Detroit, Michigan

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* Consultant to APDA. *Permanent address*: University of Michigan, Ann Arbor, Michigan.

Recursion Method for Calculating the Spatial Distribution of Resonance Absorptions

Suppose that the angular and energy distribution of neutrons is known at the surface of a slab which has a single level Breit-Wigner resonance absorption cross section and a constant scattering cross section. We wish to calculate the spatial distribution of these neutrons absorbed in their first collision within the slab.

Since the Breit-Wigner formula is symmetric about the resonance energy, only the symmetric part of the energy dependence of the surface flux will be relevant; let us assume that by some means this symmetric part has been expressed in powers of

$$\psi(x) \equiv (1 + x^2)^{-1}, \tag{1}$$

where $x \equiv 2(E - E_0)/\Gamma$ in terms of the resonance energy E_0 and half width Γ (a formulation for calculating these components directly from the transport equation is currently being attempted).

The angular dependence of each term can now be expanded in powers of μ, the cosine of the angle between the direction of flight and the normal to the slab. For each resulting component of the vector flux, $\mu^l \psi^n$, the absorption rate at a distance z within the slab is proportional to

$$A_{ln}(z) = \frac{1}{\pi}\int_0^1 d\mu \mu^l \int_{-\infty}^{\infty} dx [\psi(x)]^{n+1} \cdot \exp -\frac{z}{\mu}[\Sigma_0 \psi(x) + \Sigma_{\mathrm{s}}], \tag{2}$$

where $\Sigma_0 \psi$ and Σ_{s} are the macroscopic absorption and scattering cross sections. Of most interest is A_{00}, the absorption rate of a thin resonance when the surface flux is isotropic; this case has been treated asymptotically (*1*) for zero Σ_{s}.

Rather than seeking closed form approximations, let us consider what would be practical for accurate numerical calculations. Since evaluating $A_{ln}(z)$ for all values of l, n, and z by any direct method certainly appears to be a prodigious task, it would seem worthwhile to search for simple relations between adjacent values: what is needed is a differential equation in the continuous variable z and recursion formulas in the discrete indices l and n.

Thus, let us first consider A_{l0}. By letting t in Eq. (2) of ref. *2*, p. 172 equal $2\psi - 1$, one finds that our integral over x equals $\pi f(t\rho)$, where $f(\xi) \equiv e^{-(1+\alpha)\xi} I_0(\xi)$, $\alpha \equiv 2\Sigma_{\mathrm{s}}/\Sigma_0$, $t \equiv 1/\mu$, $\rho \equiv \Sigma_0 z/2$, and I_0 is a modified Bessel function (a relationship that has also been used by several authors previously). Changing the variable of integration from μ to t gives

$$A_{l0}(\rho) = \int_1^{\infty} f(t\rho) \frac{dt}{t^{l+2}}. \tag{3}$$

Therefore,

$$\frac{dA_{l0}(\rho)}{d\rho} = \int_1^\infty f'(t\rho)\frac{dt}{t^{l+1}} = \frac{1}{\rho}\int_1^\infty \frac{1}{t^{l+1}}df(t\rho)$$

$$= \frac{1}{\rho}\left[\frac{f(t\rho)}{t^{l+1}}\right]_1^\infty + \frac{l+1}{\rho}\int_1^\infty f(t\rho)\frac{dt}{t^{l+2}} \quad (4)$$

$$= \frac{1}{\rho}\Big[(l+1)A_{l0}(\rho) - f(\rho)\Big].$$

Once $f(\rho)$ has been obtained, these simple first order differential equations can be solved numerically *for each l*. Bessel's equation can be used to derive a differential equation that provides a rapid numerical method for determining $f(\rho)$:

$$f''(\rho) + \left(2 + 2\alpha + \frac{1}{\rho}\right)f'(\rho) + \left[\alpha(2+\alpha) + \frac{(1+\alpha)}{\rho}\right]f(\rho) = 0. \quad (5)$$

For higher values of n, $A_{ln}(\rho)$ can be obtained by differentiating Eq. (2):

$$-2A_{l,n+1}(\rho) = \alpha A_{l,n}(\rho) + A'_{l+1,n}(\rho), \quad (6)$$

where, as in Eq. (5), the prime means a derivative with respect to ρ. Equation (6) can be used in conjunction with Eq. (4). For a given accuracy, the use of successively higher derivatives would, of course, severely limit the coarseness of the mesh size used, but since $A_{ln}(\rho)$ vanishes very rapidly with ρ as n increases, high accuracy here is not really required.

All that remains to be done is to find starting and possible ending solutions for Eqs. (4) and (5). We note first of all that the homogeneous solution to Eq. (4) is ρ^{l+1}, which becomes infinite for large ρ and, therefore, must have a zero coefficient. For small ρ, multiplying the power series expansions for $e^{-(1+\alpha)\rho}$ and $I_0(\rho)$ together gives

$$f(\rho) = 1 - (1+\alpha)\rho + [\tfrac{1}{2}(1+\alpha)^2 + \tfrac{1}{4}]\rho^2 \cdots . \quad (7)$$

The power series expansion for $A_{l0}(\rho)$ can be obtained by inserting Eq. (7) into Eq. (4) and equating the coefficients of equal powers of ρ:

$$A_{l0}(\rho) = B_{l+1}(\rho) - (1+\alpha)B_l(\rho)\rho + [\tfrac{1}{2}(1+\alpha)^2 + \tfrac{1}{4}]B_{l-1}(\rho)\rho^2 \cdots , \quad (8)$$

where the functions $B_l(\rho)$ are defined:

$$B_l(\rho) = 1/l \quad \text{if} \quad l \neq 0$$
$$= -ln\rho \quad \text{if} \quad l = 0.$$

For large ρ, the properties of $I_0(\rho)$ tell us that

$$f(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{-\alpha\rho}\left[1 + \frac{1}{8\rho} + \frac{9}{128\rho^2} + \cdots\right]. \quad (9)$$

By writing

$$A_{l0}(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{-\alpha\rho}\left[a + \frac{b}{\rho} + \frac{c}{\rho^2} + \cdots\right], \quad (10)$$

inserting the expression in Eq. (4), and equating coefficients of equal powers of ρ, we obtain:

$$a = 0,\ b = 1/\alpha,\ c = 1/(8\alpha) - (l+\tfrac{5}{2})/\alpha^2 \quad \text{if} \quad \alpha\rho \gg 1,$$

$$a = 1/(l+\tfrac{3}{2}),\ b = (\tfrac{1}{8})/(l+\tfrac{5}{2}),\ c = (\tfrac{9}{128})/(l+\tfrac{7}{2}) \quad \text{if} \quad \alpha = 0. \quad (11)$$

Finally, it should be mentioned that if $f(\xi)$ is generalized to a form that allows $\psi(x)$ to be the Doppler broadened resonance function, Eqs. (3, 4, and 6) will still be valid.

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HARVEY J. AMSTER
Department of Nuclear Engineering
University of California
Berkeley, California
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Treatment of Annular Voids in Diffusion Theory

The usual one-dimensional multigroup diffusion codes can be used to calculate configurations with annular void regions by assigning a fictitious diffusion coefficient to the void region. The recipe is suitable for both infinite cylinders and spheres. Basically, the method is to calculate a diffusion coefficient for the void region which preserves the annular void boundary conditions consistent with the neutron streaming problem in the P-1 approximation.

In the cylindrical case, let us assume that the void region, $r_1 \leqq r \leqq r_2$, is characterized by some value of $\kappa^2 = \Sigma_a/D$ where $\Sigma_a \ll D$. Diffusion theory gives

$$\phi_v(r) = AI_0(\kappa r) + BK_0(\kappa r) \qquad r_1 \leqq r \leqq r_2 \quad (1)$$

where $\phi_v(r)$ is the void neutron flux and I_0 and K_0 are the modified Bessel functions. For $\kappa r \ll 1$, Eq. (1) becomes

$$\phi_v(r) = A + B\ln\kappa r. \quad (2)$$

By differentiation, B can be expressed as $r(\partial\phi_v/\partial r)$. Using the continuity of the neutron current across the interfaces at r_1 and r_2, we have

$$B = \frac{D_1 r_1}{D}\left(\frac{\partial\phi}{\partial r}\right)_1 = \frac{D_2 r_2}{D}\left(\frac{\partial\phi}{\partial r}\right)_2 \quad (3)$$

where $D_1(D_2)$ represents the diffusion coefficient for the region $r < r_1 (r > r_2)$, D is the void "diffusion coefficient" and the derivatives are evaluated for the diffusing regions at the void interfaces. Using Eqs. (2) and (3) and assuming continuity of the neutron flux at the interfaces gives

$$\phi(r_1) - \phi(r_2) = -\frac{D_1 r_1}{D}\left(\frac{\partial\phi}{\partial r}\right)_1 \ln\left(\frac{r_2}{r_1}\right). \quad (4)$$

The P-3 approximation to the neutron streaming problem across an annular void in cylindrical geometry has been derived by Tait (*1*). Using only the P-1 terms and assuming the diffusion approximation, i.e., that the net neutron cur-