Solution of the transport equation by S_n approximations. LA-1891 (1955).

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Slowing Down with Anisotropic Scattering

The technique of expanding Laplace transforms in a series of the transform variable before inversion was applied in an earlier paper (*1*) to determine the flux obtained from an energy distributed neutron source in an infinite homogeneous system. This method is capable of extension to slowing down when the differential scattering cross section of the nuclei is anisotropic in the center of mass system. We will assume that the angular distribution of scattered neutrons is independent of the initial energy and will also omit extraneous complications, such as absorption, in order to give a simple outline of the main feature of the problem.

The angular distribution of scattered neutrons is assumed to be given by $g(\mu)$ where μ is the cosine of θ, the scattering angle in the center of mass system. In terms of the initial and final lethargies u' and u,

$$\mu = \frac{2}{1-\alpha} e^{-(u-u')} - \frac{1+\alpha}{1-\alpha},$$

where α is the maximum fractional energy loss on a collision.

The neutron balance equation in the lethargy interval du at u is

$$\Sigma_s(u)\phi(u) = \int_{u-u_m}^{u} \Sigma_s(u')\phi(u')\, e^{-(u-u')} \cdot g(\mu) \frac{du'}{1-\alpha} + S(u) \tag{1}$$

where $\phi(u)$ is the flux, $\Sigma_s(u)$ the scattering cross section, and $S(u)$ the source, all at lethargy u, and $u_m = \ln(1/\alpha)$. Taking the Laplace transform of Eq. (1) and rearranging terms we find

$$\mathcal{L}\{\Sigma_s(u)\phi(u)\} = \frac{1}{1-K(p)}\, \mathcal{L}\{S(u)\} \tag{2}$$

where

$$\begin{aligned} K(p) &= \int_0^{u_m} e^{-(p+1)u} g\left(\frac{1}{\delta} e^{-u} - \frac{1}{\delta} + 1\right) \frac{du}{1-\alpha} \\ &= \frac{1}{2}\int_{-1}^{1} [1-\delta(1-\mu)]^p g(\mu)\, d\mu \end{aligned} \tag{3}$$

and $\delta = \frac{1}{2}(1-\alpha)$.

The normalizing condition on $g(\mu)$ is $\int_{-1}^{1} g(\mu)\, d\mu = 2$ which is equivalent to $K(0) = 1$. Thus the expansion of $\{1-K(p)\}^{-1}$ in powers of p is

$$\frac{1}{1-K(p)} = \frac{1}{\xi_A\, p} + \gamma_{0A} + \gamma_{1A}\, p + \gamma_{2A}\, p^2 + \cdots$$

where ξ_A and γ_{nA} are constants to be determined.

Inverting Eq. (2) gives

$$\Sigma_s(u)\phi(u) = \frac{1}{\xi_A}\int_0^u S(u')\, du' + \gamma_{0A}\, S(u) + \gamma_{1A}\, S'(u) + \cdots. \tag{4}$$

The coefficients in Eq. (4) are found from the relation

$$\gamma_{nA} = \lim_{p\to 0} \frac{1}{p^n}\left[\frac{1}{1-K(p)} - \frac{1}{\xi_A\, p} - \gamma_{0A} - \gamma_{1A}\, p - \cdots - \gamma_{n-1,A}\, p^{n-1}\right]$$

with

$$\frac{1}{\xi_A} = \lim_{p\to 0}\frac{p}{1-K(p)} = -\frac{1}{K'(0)}.$$

Thus

$$\gamma_{nA}\{K'(0)\}^2 = \frac{K^{(n+2)}(0)}{(n+2)!} - K'(0)\left[\frac{\gamma_{0A}\, K^{(n+1)}(0)}{(n+1)!} + \frac{\gamma_{1A}\, K^{(n)}(0)}{n!} + \cdots + \frac{\gamma_{n-1,A}\, K''(0)}{2!}\right] \tag{5}$$

where from equation (3) we have that

$$K^{(r)}(0) = \frac{1}{2}\int_{-1}^{1} \{\ln[1-\delta(1-\mu)]\}^r g(\mu)\, d\mu. \tag{6}$$

The logarithm in the integrand of (6) can be expanded in powers of δ to any required accuracy, thereby determining the coefficients in Eq. (4) for any scattering material. For all but the lightest nuclei it will be found adequate to retain only the terms of $O(\delta^2)$ in Eq. (6) and the first two terms of the series in Eq. (4).

As an illustration we now treat the simplest problem of a heavy scatterer for which $\alpha \to 1$. We may therefore neglect terms $O(\delta^2)$ and take $\delta = \xi$, the average logarithmic energy decrement for isotropic scattering.

Equation (6) now gives

$$\begin{aligned} K^{(r)}(0) &= (-1)^r \frac{1}{2}\xi^r \int_{-1}^{1} (1-\mu)^r g(\mu)\, d\mu \\ &= (-1)^r \xi^r \overline{(1-\mu)^r} \end{aligned} \tag{7}$$

where the bar denotes the mean value. Substituting from (7) into (5) and rearranging the terms,

$$\gamma_{nA}\overline{(1-\mu)} - \xi\gamma_{n-1,A}\frac{\overline{(1-\mu)^2}}{2!} + \cdots + (-1)^n \xi^n \gamma_{0A}\frac{\overline{(1-\mu)^{n+1}}}{(n+1)!} = (-1)^n \frac{\xi^n}{(n+2)!}\frac{\overline{(1-\mu)^{n+2}}}{\overline{(1-\mu)}}. \tag{8}$$

For a heavy scatterer the first two coefficients in the expansion (4) are thus

$$\begin{aligned} \xi_A &= \xi(1-\bar{\mu}) \\ \gamma_{0A} &= \frac{1-2\bar{\mu}+\overline{\mu^2}}{2(1-\bar{\mu})^2} \end{aligned} \tag{9}$$

where ξ_A can be shown to be the average logarithmic energy loss and $\xi_A \gamma_{0A}$ is the modified Greuling-Goertzel correction. The first of Eqs. (9) has been derived for a particular angular distribution by Weinberg and Wigner (*2*).

REFERENCES

1. A. Keane, *Nuclear Sci. and Eng.* **10,** 117 (1961).
2. A. M. Weinberg and E. P. Wigner, "Physical Theory of Neutron Chain Reactors." Univ. of Chicago Press, 1958.

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