suming and expensive. Thus it is not practical to extend the integral method to coins of very large radii.

The above comments should not be taken to indicate that there is little agreement between the two methods. Figure 1 shows that there is a large area of agreement between the two methods even when applied to detectors in water.

It should be noted that for the data in Fig. 1 scatter was assumed isotropic (i.e. $\bar{\mu} = 0$) since most of the integral calculations were carried out on this basis. Calculations for a gold coin (of 5 mils thickness and 0.5 cm radius) in water to investigate the effect of anistropy of scatter in water have been made using both methods. The results are as follows:

Average Scalar Flux in the Detector

	$\bar{\mu}$	0.0	0.3
$\bar{\phi}/\phi_0$	(variational)	0.773 ± .005	0.803 ± .005
$\bar{\phi}/\phi_0$	(integral)	0.813 ± .002	0.831 ± .002

In view of the uncertainty in reading the graphs for use in the variational method and finiteness of the numerical integrations of the integral method, the agreement as to the sign and magnitude of the effect of anistropy is quite encouraging. It is, however, unfortunate that the comparison was made in a region of small radii coins where the two methods do not agree too well in absolute magnitude.

Finally it should be emphasized that all the dimensionless plots in the second paper, i.e., Figs. 7–10 and 13–18, are not rigorously correct. They result from a compromise of about one percent between the dimensionless plots for gold and indium. Further investigation showed that the same dimensionless plots could be used for detector absorption cross section between 1.0 and 10.0 if one requires no more than plus or minus two or three percent accuracy. If, however, (1) high accuracy is required, (2) detector absorption is not considerably greater than its scatter, (3) scatter in the external medium is not isotropic, and (4) the detector is not in the size range covered, then one should return to the computer and calculate the particular cases of interest.

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The Milne Problem with a Polynomial Source

The Milne problem with a source of the form x^n has been treated by a number of authors. Lundquist and Horak have expressed the emergent flux in terms of a recursion relation (*1*). Ueno has used the probabilistic approach to obtain the emergent flux in closed form (*2*). Busbridge has derived both the emergent flux and the angle integrated flux in the interior (*3*). The latter is obtained from an iteration procedure which is shown to converge to the correct solution. The purpose of this note is to derive closed expressions for the emergent angular distribution and the total flux by using a method described in ref. *4*.

In plane geometry the energy independent transport equation for isotropic scattering in the laboratory system is

$$\mu \frac{d\psi}{dx} + \psi(x,\mu) = \tfrac{1}{2}\omega \int_{-1}^{1} \psi(x,\mu')\, d\mu' + \frac{1}{2\Sigma} Q(x), 0 \leqq \omega < 1, \quad (1)$$

where $\omega = \Sigma_s/\Sigma$, μ is the direction cosine with the positive x-axis, and $Q(x)$ is a volume source. x is measured in terms of the total mean free path. For a source of the form $Q(x) = \exp sx$ the angular distribution $\psi(0, -\mu)$, $(0 \leqq \mu \leqq 1)$, of neutrons emerging from the surface $x = 0$ of a semi-infinite slab can be shown to equal (*2, 4*)

$$\psi(0, -\mu) = (1/2\Sigma)H(-1/s)H(\mu)(1 - \mu s)^{-1}, \quad \mu \geqq 0, \quad (2)$$

where $H(\mu)$ satisfies the integral equation

$$H(\mu) = 1 + \tfrac{1}{2}\omega\mu H(\mu) \int_0^1 H(\mu')(\mu + \mu')^{-1} d\mu', \quad 0 \leqq \mu \leqq 1. \quad (3)$$

The H-functions have been discussed extensively (*3, 5, 6*). They are tabulated in the range $0 < \omega \leqq 1$ for small increments of μ (*5, 7*). Their moments, defined as

$$h_n = \int_0^1 \mu^n H(\mu)\, d\mu, \quad (4)$$

are tabulated for $0 \leqq n \leqq 20$ (*7*).

Expressing the source in terms of its Laplace transform,

$$Q(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{Q}(s) e^{sx}\, ds, \quad (5)$$

one finds for the angular distribution, according to Eq. (2),

$$\psi(0, -\mu) = \frac{1}{2\Sigma} H(\mu) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{Q}(s) H\left(-\frac{1}{s}\right) \cdot (1 - \mu s)^{-1}\, ds, \quad \mu \geqq 0. \quad (6)$$

where the contour must correspond to that of Eq. (5).

Equation (6) is quite general and can be applied to an arbitrary source $Q(x)$. Its use will be illustrated by applying it to a source of the form $Q(x) = Qx^n$. Inserting the Laplace transform of the source into Eq. (6) gives,

$$\psi^{(n)}(0, -\mu) = \frac{n!Q}{2\Sigma} H(\mu) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{H(-1/s)}{s^{n+1}(1 - \mu s)}\, ds. \quad (7)$$

If the contour is closed by an arc of radius R in the left-hand plane, the contribution of the integral along the arc will vanish as $R \to \infty$. $H(-1/s)$ and $1/(1 - \mu s)$ are regular in the left-hand plane, so that the only singularities enclosed by the contour are at $s = 0$, where the integrand has a pole of order $n + 1$. The residue at $s = 0$ is therefore

$$A_{-1} = \frac{1}{n!}\left[\frac{d^n}{ds^n}\frac{H(-1/s)}{1-\mu s}\right]_{s=0}. \tag{8}$$

The derivative can be evaluated by using Eq. (3) to express the argument in the form

$$H\left(-\frac{1}{s}\right)(1-\mu s)^{-1} = \left[(1-\mu s)\left(1 - \frac{1}{2}\omega\int_0^1 H(\mu')(1-\mu' s)^{-1}\,d\mu'\right)\right]^{-1}.$$

For small s this may be expanded into a power series,

$$H\left(-\frac{1}{s}\right)(1-\mu s)^{-1} = (1-\omega)^{-1/2}\sum_{n=0}^{\infty} a_n(\mu)s^n, \tag{9}$$

where

$$a_0 = 1, \qquad a_n = \begin{vmatrix} b_1 & -1 & 0 & \cdots & 0 \\ b_2 & b_1 & -1 & \cdots & 0 \\ \cdots & & & & \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \end{vmatrix}, \quad n \geqq 1, \tag{9a}$$

$$b_1 = \mu + gh_1, \qquad b_m = g(h_m - \mu h_{m-1}), \qquad m \geqq 2,$$

$$g = \tfrac{1}{2}\omega(1-\omega)^{-1/2}.$$

Using Eqs. (8) and (9) and substituting into Eq. (7) gives the result,

$$\psi^{(n)}(0, -\mu) = (n!)gQ(1/\omega\Sigma)H(\mu)a_n(\mu). \tag{10}$$

In order to find the space dependence of the angle integrated flux one transforms Eq. (1) into an integral equation by integrating over x from 0 to ∞. The result is,

$$\psi(0, -\mu) = \frac{\omega}{2\mu}\int_0^\infty e^{-x/\mu}\psi(x)\,dx + \frac{1}{2\Sigma\mu}\int_0^\infty e^{-x/\mu}Q(x)\,dx, \qquad \mu \geqq 0, \tag{11}$$

where $\psi(x) = \int_{-1}^{1}\psi(x, \mu)\,d\mu$ is the flux one wishes to determine. If $1/\mu$ is replaced by s, the two terms on the right of Eq. (11) become proportional to the Laplace transforms of the flux and the source respectively. Solving for the transform of the flux and inverting one finds,

$$\psi(x) = \frac{2}{\omega}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{s}\psi\left(0, -\frac{1}{s}\right)e^{sx}\,ds - \frac{1}{\omega\Sigma}Q(x). \tag{12}$$

For a source of the form Qx^n one may substitute for $\psi(0, -1/s)$ from Eq. (10) and obtain,

$$\psi^{(n)}(x) = \frac{n!\,Q}{\omega\Sigma(1-\omega)^{1/2}}\frac{1}{2\pi i}\cdot\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{s}H\left(\frac{1}{s}\right)a_n\left(\frac{1}{s}\right)e^{sx}\,dx - \frac{Q}{\omega\Sigma}x^n. \tag{13}$$

The details of integrating expressions similar to the above are discussed in ref. 4 and will not be repeated here. The contour must be closed in the left half-plane, where the contribution along the infinite arc vanishes. $H(1/s)$ has a simple pole on the real axis at $s = -k$, $0 < k \leqq 1$, but is otherwise regular in the plane cut between -1 and $-\infty$. The constant k is the positive solution of the equation

$$\frac{\omega}{2k}\ln\frac{1+k}{1-k} = 1.$$

A comparison with Eq. (9a) shows that $a_n(1/s)$ can be written in the form,

$$a_n\left(\frac{1}{s}\right) = \sum_{m=0}^{n} c_{n-m}\,s^{-m}, \tag{14}$$

where

$$c_0 = 1, \qquad c_m = \begin{vmatrix} gh_1 & -1 & 0 & \cdots & 0 \\ gh_2 & gh_1 & -1 & \cdots & 0 \\ \cdots & & & & \\ gh_m & gh_{m-1} & gh_{m-2} & \cdots & gh_1 \end{vmatrix}, \qquad m \geqq 1. \tag{14a}$$

The integrand will therefore have poles of various orders at $s = 0$. In addition to the residues at $s = 0$ and $s = -k$ the integral must be evaluated around the cut. The final result is,

$$\begin{aligned}\psi^{(n)}(x) = {}& \frac{Q}{\Sigma(1-\omega)}x^n + \frac{n!\,Q}{\Sigma\omega(1-\omega)}\sum_{m=0}^{n-1}\frac{1}{m!}A_{n-m}\,x^m \\ & - \frac{n!\,Q}{\Sigma\omega(1-\omega)^{1/2}}\left\{\frac{a_n(-1/k)}{\left(\frac{\omega}{1-k^2}-1\right)H\left(\frac{1}{k}\right)}e^{-kx}\right. \\ & \left. + \frac{\omega}{2}\int_0^1 \frac{a_n(-\eta)e^{-x/\eta}\,d\eta}{H(\eta)\left[\left(1 - \frac{\omega\eta}{2}\ln\frac{1+\eta}{1-\eta}\right)^2 + \left(\frac{\pi\omega\eta}{2}\right)^2\right]}\right\},\end{aligned} \tag{15}$$

where

$$\begin{aligned} A_{n-m} &= \sum_{i=0}^{n-m}(-1)^i C_i C_{n-m-i} && \text{if } n - m \text{ is even,} \\ &= 0 && \text{if } n - m \text{ is odd.}\end{aligned}$$

The constants c are taken from Eq. (14a).

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