

$$ZP(j\omega) = \frac{1 - j\omega \sum_i \beta_i / (j\omega + \lambda_i)}{j\omega[l + \sum_i \beta_i / (j\omega + \lambda_i)]}$$

zero power transfer function

- β_i = fraction of neutrons that are delayed in i th group
 λ_i = decay constant for i th group
 K_{ex} = $K_{effective} - 1$ excess reactivity
 l = effective prompt neutron lifetime
 ω = frequency (rad/sec)
 $Re[ZP(j\omega)]$ = real part of $ZP(j\omega)$

A quantity of interest is the amount of K_{ex} necessary to generate a given ratio of second order harmonic to fundamental $|X_2/X_1|$. Equation (1) represents a quadratic vector equation in $K_{ex}/2$ which leads for a given ratio of "a" to two solutions. Only one solution is applicable.

$$\left(\frac{K_{ex}}{2}\right)^2 = -\frac{b}{2m} \left[\frac{1}{2} \left(\frac{4mc}{b^2}\right) + \frac{1}{8} \left(\frac{4mc}{b^2}\right)^2 + \dots \right] \quad (2)$$

where:

$$\begin{aligned} m &= a^2(\beta_x^2 + \beta_y^2) \\ b &= 2\beta_x a^2 - (\alpha_x^2 + \alpha_y^2) \\ c &= a^2 \\ \alpha &= ZP(j\omega) = \alpha_x + j\alpha_y \\ \beta &= ZP(2j\omega)ZP(j\omega) - 2ZP(j\omega) \operatorname{Re}[ZP(j\omega)] = \beta_x + j\beta_y \end{aligned}$$

The second solution has to be discarded because it leads to values of K_{ex} near prompt critical for which the assumptions made in the original derivation become invalid.

Similar to Eq. (1), we form the ratio "a" of second order harmonic and the fundamental for a reactor at power.

$$\left| \frac{X_2}{X_1} \right| = a$$

$$= \frac{LP(2j\omega)[1 + n_0PK(j\omega)LP(j\omega)]K_{ex}/2}{1 + \left\{ \begin{aligned} &LP(2j\omega)LP(j\omega) + LP(2j\omega)n_0PK(j\omega)[LP(j\omega)]^2 \\ &+ n_0PK^*(j\omega)LP^*(j\omega)LP(2j\omega) \\ &\cdot \{LP(j\omega) + n_0PK(j\omega)[LP(j\omega)]^2\} \\ &[-2LP(j\omega) \operatorname{Re}[LP(j\omega) + n_0PK(j\omega)|LP(j\omega)]^2] \end{aligned} \right\} (K_{ex}/2)^2} \quad (3)$$

where:

- P = power level in Mw
 $K(j\omega)$ = $(\nabla k/k)/Mw$ power coefficient
 n_0 = steady state flux (Since we are interested in reactivity changes with respect to the steady state flux n_0 , we are free to set $n_0 = 1$.)

$$LP(j\omega) = \frac{ZP(j\omega)}{1 - ZP(j\omega)PK(j\omega)} = \frac{\delta n}{n_0} / K_{ex} \quad (4)$$

Equation (4) follows from the following symbolism assuming linearity (2).

$$\delta k(j\omega) = K_{ex}(j\omega) + n_0PK(j\omega) \frac{\delta n(j\omega)}{n_0} \quad (5)$$

The second term in Eq. (5) assumes that changes in reactivity are linear functions of the power variation. We have written both the zero power transfer function $ZP(j\omega)$ and

* indicates conjugate complex function.

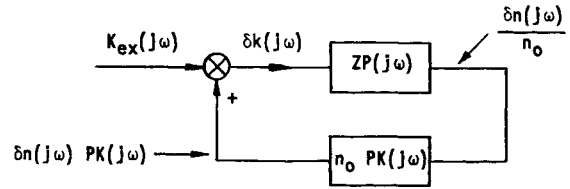


FIG. 1. Feedback system

the resulting incremental reactivity in the frequency domain and we can represent $ZP(j\omega)$ and Eq. (5) symbolically in a feedback system shown in Fig. 1 (5). In this symbolic notation the feedback is positive. This is done in order to retain the physical meaning of the sign of the power coefficient.

In an analog manner, as for the zero power case, we can solve for K_{ex} necessary to obtain a given ratio $a = |X_2/X_1|$ for a reactor under power.

For the terms in Eq. (2) we get (and again the second solution has to be discarded):

$$\begin{aligned} m &= a^2(\lambda_x^2 + \lambda_y^2) \\ b &= 2a^2\lambda_x - (\delta_x^2 + \delta_y^2) \\ c &= a^2 \\ \delta &= LP(2j\omega)[1 + n_0PK(j\omega)LP(j\omega)] \\ \lambda &= LP(2j\omega)LP(j\omega) + LP(2j\omega)n_0PK(j\omega)[LP(j\omega)]^2 \\ &\quad + n_0PK^*(j\omega)LP^*(j\omega)LP(2j\omega) \\ &\quad \cdot \{LP(j\omega) + n_0PK(j\omega)[LP(j\omega)]^2\} \\ &\quad - 2LP(j\omega) \operatorname{Re}[LP(j\omega) + n_0PK(j\omega)|LP(j\omega)]^2 \end{aligned}$$

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Simplified Calculation on Thermal Transient of a UO_2 Fuel Rod

The thermal behavior of a cylindrical UO_2 fuel rod is characterized by many parameters, namely, a high thermal resistance and a relatively small capacitance of the ceramic

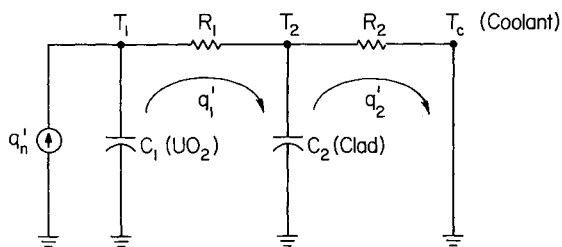


FIG. 1. Equivalent circuit of fuel rod

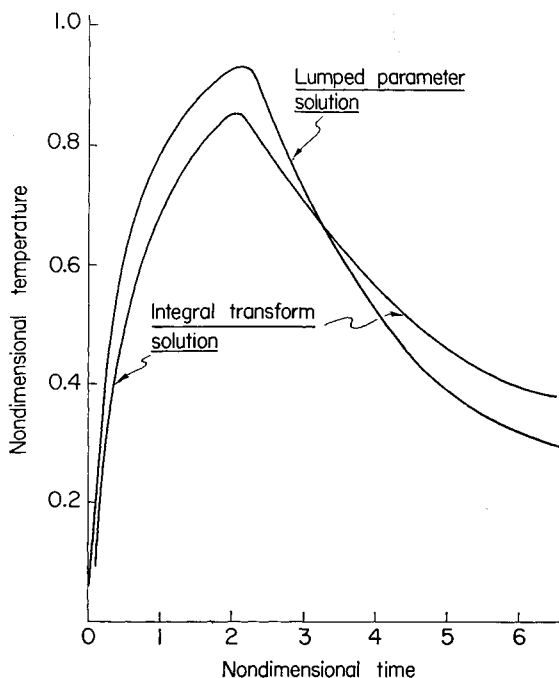


FIG. 2. Comparison of solutions (2)

UO₂ pellet, a low resistance and a nonnegligible capacitance of the metallic clad, and a contact resistance between the clad and pellet. In analyzing its thermal transient, a Hankel Transform (1) or a Finite Integral Transform (2) may be used. However, in both cases, the exact solutions are in the forms of infinite series, and it is almost impossible to make a quick evaluation by hand calculation. Hence, a lumped parameter technique was introduced for simplifying the

calculations in the UO₂ fuel rod, the lumped parameter technique gives a fairly accurate solution in comparing with other exact methods. In ref. 1, the maximum deviation in heat transfer rate was evaluated as less than 10%. In ref. 2, the maximum clad temperature obtained by present technique agreed with the solution from a Finite Integral Transform within 8% and the value from lumped parameter is higher than the exact solution. This indicates that the simplified solution is conservative. Both solutions are shown in Fig. 2. The solution from one-lumped capacitance in the pellet was also compared with that of an analog computer using the seven lumped capacitance model by Cunningham (3). He found that the maximum clad temperatures after a scram following a total loss of power to the pumps evaluated by both methods, agreed within 2%. These results of comparison indicate that a lumped parameter technique is adequate in calculating the thermal transient of a UO₂ fuel rod for engineering purposes.

The rate equations for the heat transfer of a UO₂ fuel rod (Fig. 1) can be written as:

$$q_n' = C_1 \frac{dT_1}{dt} + \frac{T_1 - T_2}{R_1} \quad (1)$$

$$\frac{T_1 - T_2}{R_1} = C_2 \frac{dT_2}{dt} + \frac{T_2 - T_c}{R_2} \quad (2)$$

where

q_n' = nuclear heating, Btu/sec-ft (of fuel rod)

q_2' = heat transfer to coolant, Btu/sec-ft

$$q_2' = (T_2 - T_c)/R_2$$

C_1 = capacitance of pellet, Btu/°F ft, $C_1 = \pi r_1^2 c_1 \rho_1$

C_2 = capacitance of clad, Btu/°F ft, $C_2 = 2\pi r_2 \Delta r c_2 \rho_2$

R_1 = resistance of UO₂ and gap, sec ft °F/Btu

$$R_1 = (1/8\pi k_1) + (1/2\pi r_1 h_g)$$

k_1 is UO₂ thermal conductivity, h_g is contact conductance.

R_2 = resistance of coolant film, sec ft °F/Btu

$$R_2 = (1/2\pi r_2 h_2)$$

T_1 = average pellet temperature, °F

T_2 = average clad temperature, °F

T_c = bulk coolant temperature, °F

After a pipe rupture, the system pressure drops with the time and so does the saturation temperature of the coolant. Hence $T_c = T_c(t)$. The time, t , is counted from the instant of rupture. By taking Laplace transformation of Eqs. (1) and (2), we get:

$$T_2(s) = \frac{R_2 q_n'(s) + (C_1 R_1 s + 1)T_c(s) + R_2 C_1 T_1(0) + R_2 C_2(R_1 C_1 s + 1)T_2(0)}{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1) s + 1} \quad (3)$$

$$T_1(s) = \frac{(R_2 C_2 s + 1 + R_2/R_1)R_1 q_n'(s) + T_c(s) + (R_2 C_2 s + 1 + R_2/R_1)R_1 C_1 T_1(0) + R_2 C_2 T_2(0)}{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1) s + 1} \quad (4)$$

calculation in which the thermal resistances and capacitances of the pellet and the clad are evaluated at their average conditions in time and space. Each quantity is lumped at the middle of its physical geometry and axial heat conduction is neglected. The equivalent thermal circuit is shown in Fig. 1.

Due to the way of distribution of the resistances and

where $q_n'(s)$ is $Lq_n'(t)$ and $q_n'(t)$ is the decay heat. For a pressurized water reactor operated for a long time, the decay heat equation can be written approximately as:

$$q_n'(t)/q_n'(0) = 0.056e^{-0.004t} + 0.20e^{-0.1t}, \quad \text{for } t \leq 60 \text{ sec and}$$

$$q_n'(t)/q_n'(0) = 0.031 + 0.025e^{-0.009t}, \quad \text{for } 60 < t \leq 350 \text{ sec}$$

By knowing the coolant temperature, $T_c = T_c(t)$, the histories of pellet temperature and clad temperature are readily computable from the inverse transformation of Eqs. (3) and (4).

In a loss-of-flow accident, a sudden decrease of the rod surface heat transfer coefficient due to the occurrence of film boiling after DNB will cause a temperature rise in the clad even though the reactor is scrammed. The magnitude of this temperature rise affects the strength of clad and thus determines the integrity of the fuel rod.

Two simplifications can be made in analysis of this accident. Firstly, when the pumps lose power, the system pressure does not change significantly and the coolant temperature, T_c , remains approximately constant. Secondly, the maximum clad temperature usually occurs within 10 sec from the instant of loss power to the pumps. Hence, the decay heat can be assumed to be constant in this short time period, i.e., $q_n'(t) = 0.22 q_n'(0)$. Then Eqs. (1) and (2) can be combined as:

$$q_n' = C_1 C_2 R_1 \ddot{q}_2' + (C_1 + C_2 + C_1 R_1 / R_2) \dot{q}_2' + q_2' \quad (5)$$

where

$$q_2' = (T_2 - T_c) / R_2$$

The boundary conditions of the above equation are:

$$\text{For } t \leq 0: q_n' = q_n'(0), \quad R_2 = R_{2,0}$$

$$\text{For } 0 < t \leq t_1: q_n' = q_n'(0), \quad R_2 = R_{2,fb}$$

$$\text{For } t \geq t_1: q_n' = 0.22 q_n'(0), \quad R_2 = R_{2,fb}$$

where

$$t_1 = t_{\text{scram}} - t_{\text{DNB}},$$

all times measured from loss of power to pumps. In order to facilitate a parameter study of the fuel rod transient behavior, these parameters were grouped by Hunin and Tong (4). They rearranged Eq. (5) and its boundary conditions into the following equation:

$$\frac{(T_{2,\text{max}} - T_c)}{q_n'(0)} = Z \left(\frac{\beta_2}{\beta_1} \right)^{\beta_1 / (\beta_1 - \beta_2)} \cdot \left(\frac{Z}{Y} \right)^{\beta_2 / (\beta_1 - \beta_2)} + 0.22 R_{2,fb} \quad (6)$$

where

$$Y = \left[R_{2,0} - R_{2,fb} - \frac{1}{\beta_2 A_c} \left(1 - \frac{R_{2,0}}{R_{2,fb}} \right) \right] e^{\beta_1 \tau_1} + R_{2,fb} (1 - 0.22)$$

$$Z = \left[\frac{\beta_1}{\beta_2} (R_{2,0} - R_{2,fb}) - \frac{1}{\beta_2 A_c} \left(1 - \frac{R_{2,0}}{R_{2,fb}} \right) \right] e^{\beta_2 \tau_1} + \frac{\beta_1}{\beta_2} R_{2,fb} (1 - 0.22)$$

$$\tau_1 = t_1 / r_1^2,$$

where r_1 is pellet radius

$$\beta_1 = r_1^2 \alpha_1, \quad \beta_2 = r_1^2 \alpha_2$$

$$A_c = C_2 / r_1^2$$

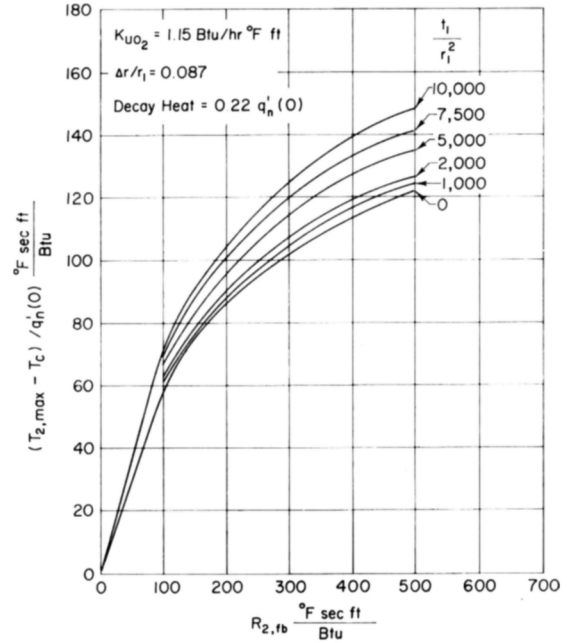


FIG. 3. Thermal transient parameters of a fuel rod

α_1 and α_2 are the two roots of the homogeneous form of Eq. (5) as

$$C_1 C_2 R_1 \alpha^2 + (C_1 + C_2 + C_1 R_1 / R_2) \alpha + 1 / R_2 = 0$$

It can be seen from Eq. (6) that the maximum clad temperature ($T_{2,\text{max}}$) is a function of τ_1 and R_2 providing the physical properties of the pellet and clad and the steady state power level ($q_n'(0)$) are known. Now we can plot the working curves to evaluate the influences of the thermal output of fuel rod, the pellet and rod sizes, the time interval between DNB and effective scram, and the film boiling heat transfer coefficient on the maximum transient clad temperature in a loss-of-flow accident. In preparing the working curves as shown in Fig. 3, the following physical parameters are assumed:

$$c_1 \text{ of fuel} = 0.08 \text{ Btu/lb } ^\circ\text{F}$$

$$\rho_1 \text{ of fuel} = 635 \text{ lb/ft}^3$$

$$c_2 \text{ of clad} = 0.16 \text{ Btu/lb } ^\circ\text{F}$$

$$\rho_2 \text{ of clad} = 474 \text{ lb/ft}^3$$

$$k_1 = \text{fuel thermal conductivity} = 1.15 \text{ Btu/hr-ft } ^\circ\text{F}$$

$$h_g = \text{gap conductivity} = 1000 \text{ Btu/hr-ft}^2 ^\circ\text{F}$$

$$h_{2,0} = \text{steady state heat transfer coefficient} = 5000 \text{ Btu/hr-ft}^2 ^\circ\text{F}$$

$$h_{2,f} = \text{heat transfer coefficient in film boiling} = 100 \text{ Btu/hr-ft}^2 ^\circ\text{F}$$

$$\Delta r = \text{clad thickness} = 0.087 r_1$$

Thermal resistance of clad was neglected.

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A Differential Equation for Calculating Doppler Broadened Resonances

A Doppler-broadened Breit-Wigner resonance is commonly approximated⁽¹⁾ as the unbroadened value at the resonance energy multiplied by

$$\psi(\beta, x) = \frac{1}{\beta\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp -[(x-y)/\beta]^2}{1+y^2} dy, \quad (1)$$

where

$$\beta = (4/\Gamma)\sqrt{E_R kT/A} \quad \text{and} \quad x = 2(E - E_R)/\Gamma. \quad (2)$$

E_R and Γ are the resonance energy and half-width; E is the laboratory-system energy of the incident neutron; kT is the energy of thermal motion of the absorber, and A is its mass number.

For the calculation of resonance integrals or detailed neutron flux, $\psi(\beta, x)$ is required for many values of x but only one value of β for each resonance. Thus while the well-known formula,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2}{\beta} \frac{\partial \psi}{\partial \beta} \quad \text{with} \quad \psi(0, x) = \frac{1}{1+x^2}, \quad (3)$$

provides an alternative to numerically integrating the expression in Eq. (1), it suffers from the fact that $\psi(\beta, x)$ can be obtained for a given β only after the complete x -dependence has been determined for all smaller values. Therefore, for most programs requiring values of ψ without recourse to tables, it would be very desirable to have a differential equation for each resonance only in the variable x , with the parameter β held constant. Such an equation would enable one to calculate ψ entirely from those adjacent values that are needed anyway.

One can verify, after considerable manipulation, that $\psi(\beta, x)$ in Eq. (1) satisfies the simple linear second order differential equation,

$$\frac{1}{4}\beta^4 \psi'' + \beta^2 x \psi' + (1 + \frac{1}{2}\beta^2 + x^2)\psi = 1, \quad (4)$$

where the primes denote total derivatives with respect to x . To make the definition complete, there are the boundary conditions,

$$\begin{aligned} \psi(\beta, 0) &= (\sqrt{\pi}/\beta) \exp(\beta^{-2}) [1 - \text{erf}(\beta^{-1})] \\ \text{and } \psi'(\beta, 0) &= 0 \end{aligned} \quad (5)$$

for starting at the resonance energy, and $\psi(\beta, x) \simeq (1+x^2)^{-1}$ or other asymptotic expressions for starting at energies far from E_R . The present author and K. W. Morton at Harwell have both derived Eqs. (4) and (5) independently a few years ago as incidental subjects in larger technical reports. These formulas have been found very useful in a variety of codes using several of the usual numerical methods for solving second order differential equations.

If $\psi(\beta, x)$ is desired for a range of values of β as well as x , substituting the left side of Eq. (3) for ψ' in Eq. (4) will result in a more useful expression than Eq. (3) alone, since no derivatives higher than the first appear. By differentiating Eq. (4) twice with respect to x , eliminating the third derivative terms between the third and fourth order differential equations, and substituting the first derivative term of the resulting expression into Eq. (4), one can obtain a fourth order differential equation in x with only even derivatives present. Eq. (3) can then be applied to get a second order total differential equation only in the variable β (3). Sometimes the quantity

$$\varphi(\beta, x) = \frac{1}{\beta\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{y \exp -[(x-y)/\beta]^2}{1+y^2} dy, \quad (6)$$

is desired to account for Doppler broadening the interference term in resonance scattering. This quantity can be evaluated conveniently in conjunction with Eq. (4) by means of the expression⁽²⁾,

$$\varphi(\beta, x) = \frac{1}{2}\beta^2 \frac{\partial \psi(\beta, x)}{\partial x} + x \psi(\beta, x). \quad (7)$$

It would seem that the mesh spacing in x should be small compared to β in forward or central difference schemes for solving Eq. (4), since both derivative terms vanish with β . The danger of an indeterminacy would be absent if Eq. (4) were applied at one mesh point with its derivatives calculated from previous mesh points.

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Crystal Spectrometer Measurement of the MITR Thermal Neutron Spectrum*

A neutron spectrum in the wavelength range $4 \text{ \AA} > \lambda > 0.65 \text{ \AA}$ ($0.005 \text{ eV} < E < 0.2 \text{ eV}$) has been measured using a crystal

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