LETTERS TO THE EDITOR

Milne's Problem for a Velocity-Dependent Mean Free Path*

Most of the rigorous results in neutron transport theory have been obtained in the constant cross-section approximation (*1*). These results are usually assumed to apply to thermal neutrons if an appropriate average of the cross section over a Maxwellian velocity distribution is taken. The author has recently considered the problem of the decay of a fully thermalized pulse of initially fast neutrons (*2*), paying particular attention to a rigorous treatment of the neutron energy spectrum. In that work the most important question remaining unresolved was the appropriate treatment of the spatial boundary conditions in the case that the mean free path varies with neutron energy. This question can be of some practical importance in determining the geometric buckling for hydrogenous moderators, where the thermal neutron cross section is strongly energy dependent.

In the present note we consider the problem of the extrapolation distance for an energy-dependent mean free path. We consider the idealized situation of an infinite half-space with no absorption or sources and isotropic scattering. The energy transfer between neutrons and moderator is explicitly included. This problem has been solved exactly in the constant cross section approximation (*3*). There is also available a variational solution for constant cross section which gives the extrapolation distance very accurately using the solution far from the boundary as a trial function (*4*). In the following, we present an extension of this variational solution to the multivelocity problem appropriate for varying mean free path.

We consider the flux $f(z, E, \mu)$ as a function of position z, kinetic energy E, and direction $\mu = \cos^{-1}(\hat{v}\cdot\hat{z})$. The half-space for $z > 0$ contains moderating material with an isotropic energy transfer cross seciton $\Sigma(E \rightarrow E')$, and a mean free path $l(E)$ determined by

$$\int_0^{\infty} \Sigma(E\rightarrow E')\, dE' = \frac{1}{l(E)}$$

The appropriate form of the transport equation for this problem is

$$\mu \frac{\partial f}{\partial z} + \frac{1}{l(E)} f = \frac{1}{2} \int_0^{\infty} dE' \int_{-1}^{1} d\mu' \Sigma(E'\rightarrow E) f(z, E', \mu') \quad (1)$$

subject to the boundary condition

$$f(0, E, \mu) = 0 \quad \text{for} \quad \mu > 0$$

We will frequently make use of the fact that the energy transfer cross section obeys the detailed balance condition (*5*)

$$\Sigma(E' \rightarrow E)M(E') = \Sigma(E \rightarrow E')M(E)$$

where

$$M(E) = E/(kT)^2 \exp(-E/kT)$$

is the Maxwellian flux distribution at the moderator temperature T.

In the special case that the mean free path is constant, the neutron spectrum will be Maxwellian throughout the moderating material. This can be verified by substituting a solution of the form $\phi(z, \mu)M(E)$ into the transport equation and using the detailed balance condition. The function $\phi(z, \mu)$ satisfies the one-velocity transport equation

$$\mu \frac{\partial \phi}{\partial z} + \frac{1}{l}\phi = \frac{1}{2l} \int_{-1}^{1} d\mu' \phi(,z\, \mu')$$

with the boundary condition

$$\phi(0, \mu) = 0 \quad \text{for} \quad \mu > 0$$

If $l(E)$ is not constant, the spatial and energy dependence of $f(z, E, \mu)$ are not separable, and the problem becomes considerably more complicated. The variational solution, introduced by LeCaine for the one-velocity problem, however, can be generalized to the multivelocity problem defined by Eq. (1).

The total flux

$$f_0(z, E) = \int_{-1}^{1} d\mu f(z, E, \mu)$$

satisfies the integral equation[1]

$$f_0(z, E) = \frac{1}{2} \int_0^{\infty} dz' \int_0^{\infty} dE' E_1 \left(\frac{|z-z'|}{l(E)}\right) \Sigma(E'\rightarrow E) f_0(z', E')$$

Far from the boundary $f_0(z, E)$ will have the asymptotic form $(z + z_0)M(E)$. The quantity z_0 is the extrapolation distance that we wish to calculate. In order to obtain a variational expression for z_0, we first need to derive some auxiliary relations. The derivations are in a straightforward analogy with the one-velocity problem and will not be reproduced here. A more detailed discussion is given in the informal report, GAMD-944, which is available on request from the author.

We introduce the function

$$q(z, E) = f_0(z, E) - z\, M(E)$$

which satisfies the integral equation (*6*)

* Work supported by the U. S. Atomic Energy Commission under contract #AT(04-3)-167.

[1] The functions $E_n(x)$ are the same as in references 1, 3, and 4.

$$q(z, E) = \frac{1}{2} l(E)M(E)E_3\left(\frac{z}{l(E)}\right) + \frac{1}{2}\int_0^\infty dz' \int_0^\infty dE' E_1\left(\frac{|z-z'|}{l(E)}\right)\Sigma(E'\to E)q(z', E') \tag{2}$$

In terms of $q(z, E)$ the extrapolation distance is given by

$$z_0 = \frac{3}{2\bar{l}}\left[\frac{\overline{l^2}}{4} + \frac{I_1}{2}\right]$$

where

$$I_1 = \int_0^\infty\int_0^\infty\int_0^\infty dE\, dE'\, dz l(E)E_3\left(\frac{z}{l(E)}\right)\Sigma(E'\to E)q(z, E') \tag{3}$$

$$\bar{l} = \int l(E)M(E)\, dE$$

and

$$\overline{l^2} = \int l^2(E)M(E)\, dE$$

By use of the integral equation (2) satisfied by $q(z, E)$, and the detailed balance condition, it can be shown that a stationary expression for I_1 is given by

$$'I_1' = \tilde{I}_1^2/\tilde{I}_2 \tag{4}$$

where

$$\tilde{I}_2 = \int_0^\infty\int_0^\infty\int_0^\infty dE\, dE'\, dz\tilde{q}(z, E)\Sigma(E'\to E)[M(E)]^{-1} \times\left[\tilde{q}(z, E') - \frac{1}{2}\int_0^\infty\int_0^\infty dE'' dz' E_1\left(\frac{|z-z'|}{l(E')}\right)\Sigma(E''\to E')\tilde{q}(z', E'')\right]$$

and $\tilde{I}_1$ is the integral given in (3) with $q(z, E)$ replaced by the trial function $\tilde{q}(z, E)$. If $\tilde{q}(z, E)$ is chosen as the solution of (2), the expression (4) for I_1 will be maximized.

The simplest trial function is the asymptotic solution $\tilde{q}(z, E) = CM(E)$. The integrals for this trial function are the same that occur in the one-velocity problem and give

$$z_0 = \frac{3}{8}\frac{\overline{l^2}}{\bar{l}} + \frac{1}{3}\bar{l} \tag{5}$$

For constant mean free path this reduces to 0.7083 l, which is only 0.3% smaller than the exact value. The prescription given by (5) is, therefore, likely to be quite good if the variation of the mean free path over the thermal energy region is not too great.

It is of some interest to compare the extrapolation distance and the thermal diffusion coefficient in a purely thermal neutron spectrum. For isotropic scattering the thermal diffusion coefficient is given by $D = \bar{l}/3$. For constant mean free path, we therefore have $z_0 = 2.13D$. As an example of a varying mean free path we consider the case of water where the transport cross section varies approximately as $1/v$ (*9*). We will assume that the isotropic scattering results can be carried over by replacing $l(E)$ by the transport mean free path. [This is known to be accurate for the diffusion coefficient (*2*), and for the extrapolation distance in the case of constant cross section (*1*).] The relation between extrapolation distance and diffusion coefficient for a $1/v$ cross section, as obtained from (5), is

$$z_0 = 2.13D\left[\frac{8}{17} + \frac{9}{17}\left(\frac{32}{9\pi}\right)\right]$$

The extrapolation distance for water is, therefore, approximately 7% greater than would be obtained from the diffusion coefficient and the assumption of constant cross section.

There is, however, a great deal of the physics of the problem omitted by the use of the asymptotic solution as trial function. No account is taken of the departure from an equilibrium velocity distribution near the boundary. This departure is caused by the preferential leakage of neutrons of longer mean free path, and depends on the ability of collisions with the moderator to restore equilibrium. These effects can be considered, in principle, by using a trial function which goes to the correct asymptotic form at large distances but includes a non-Maxwellian transient near the boundary. At this point the analogy with the constant cross section case breaks down, and the evaluation of the integrals becomes extremely complicated. Since the results to date were felt to be of sufficient interest to be put on record, this letter has been written.

1. B. Davisson, "Neutron Transport Theory," Oxford Univ. Press, London and New York, 1957.
2. M. Nelkin, *Nuclear Sci. and Eng.* **7,** 210 (1960).
3. G. Placzek and W. Seidel, *Phys. Rev.* **72,** 550 (1947).
4. J. Le Caine, *Phys. Rev.* **72,** 564 (1947).
5. Hurwitz, Nelkin, and Habetler, *Nuclear Sci. and Eng.* **1,** 280 (1956). The detailed balance condition is discussed in appendix A.
6. All integrations involving E_n functions carried out in this report are the same as in the one velocity case. A table of integrals involving $E_n(x)$ is available in the Canadian report MT-131 by J. Le Caine.
7. P. F. Zweifel and C. D. Petrie, KAPL-1469.

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Received November 10, 1959

Volcanic Energy Resources

One of the possibilities for long term world energy production is the earth's internal heat. Relatively little is known of its total reserves, or even what fraction is being replenished by radioactivity. An estimate can be made, however, of the average amount of energy being wasted annually through the high-temperature process of lava emission. This may be only a small fraction of the total energy production. A value of 0.8 km^3 of lava per year (Sapper 1927) is still considered (*1*) the best estimate, based on production since 1500 A.D. Taking an estimate of 2000°F at emission, a density of 3, and an average atomic weight of