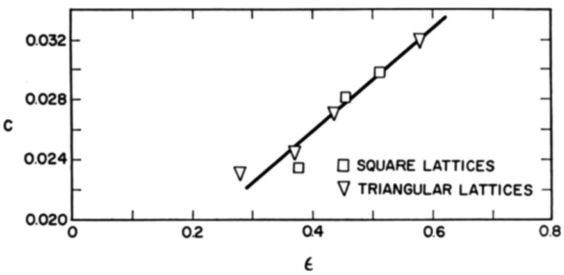

FIG. 2.

The method of least squares has been applied to the data available for the fully turbulent region, and the value of the coefficient, C, of Eq. (2) determined for each of the lattices investigated. As can be seen by the graphical representation of the results in Fig. 1, C appears to vary linearly with the S/D ratio. At the same S/D values, square pitch lattices, which are more open, give higher values of C than do the triangular pitch lattices. At Reynolds numbers between 2.5×10^4 and 10^6 we have for triangular pitch lattices, where S/D lies between 1.1 and 1.5

$$C = 0.026\ (S/D) - 0.006 \tag{3}$$

and for square pitch lattices, where S/D lies between 1.1 and 1.3

$$C = 0.042\ (S/D) - 0.024 \tag{4}$$

It is instructive to compare the results for square and triangular pitch lattices when plotted as a function of ϵ, the ratio of the water flow area to the total cross sectional area of an infinite lattice. As can be seen from Fig. 2, both lattice types yield essentially the same heat transfer coefficients at equivalent values of ϵ.

It should be noted that for almost all cases of interest, Eqs. (3) and (4) yield higher heat transfer coefficients than predicted by the Colburn equation. This should be an aid to the reactor designer since somewhat lower fluid velocities can now be used to obtain the high heat transfer coefficients desired.

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The Double Spherical Harmonic Method for Cylinders and Spheres

Several attempts to extend the double spherical harmonics method of Yvon (*1*) to cylindrical and spherical systems have appeared in the unpublished reactor technology literature. Different sets of differential equations for the same system have been suggested depending on the treatment of a product of singular functions which occurs in the analysis.

In this note we would like to point out that when one attempts to use the Yvon method for cylinders or spheres, one encounters the problem of finding the product $Y \cdot \delta$ of a Dirac δ function and a Heaviside step function, Y, that is, $Y(x) = 1$ for $x > 0$, $Y(x) = 0$ for $x < 0$. It is well known that even if one interprets these functions as distributions in the sense of L. Schwartz (*2*), the product $Y \cdot \delta$ is not defined. However, it is possible to make use of a product of distributions defined by H. Koenig (*3*) to obtain double spherical harmonics moment equations for the cylinder and sphere. The distribution product of $Y \cdot \delta$ defined by Koenig involves an arbitrary constant which must be determined by physical considerations. The same result, still involving an arbitrary constant, can be obtained without explicit use of distribution theory.

The one-velocity transport equation for a system with cylindrical symmetry may be written

$$\sin\theta\,[\cos\phi\,\frac{\partial}{\partial r}\,f(r,\theta,\phi) - \frac{\sin\phi}{r}\,\frac{\partial}{\partial\phi}\,f(r,\theta,\phi)] + \Sigma f(r,\theta,\phi)$$

$$= \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta'\,\frac{\Sigma_S}{4\pi}\,(r, \cos\theta_0)\,f(r,\theta'\phi')\sin\theta' + S'\,(r,\theta,\phi)$$

We expand the scattering kernel in ordinary spherical harmonics and for simplicity keep only the first term corresponding to isotropic scattering. To expand the flux in the double P_1 approximation we start with the usual first two spherical harmonics required for a symmetrical solution, i.e., 1 and $\sin\theta\cos\phi$, and construct the corresponding non-orthogonal set of "double spherical harmonics"

$$F_1 = (2\pi)^{-1/2}A, \qquad F_2 = (2\pi)^{-1/2}B,$$
$$F_3 = \sin\theta\cos\phi\,A, \qquad F_4 = \sin\theta\cos\phi B$$

where

$$A = 1 \text{ for } -\pi/2 < \phi < \pi/2, \qquad 0 \text{ otherwise}$$
$$B = 1 \text{ for } \pi/2 < \phi < 3\pi/2, \qquad 0 \text{ otherwise.}$$

We can write A and B using Heaviside step functions as

$$A = Y_{-\pi/2} - Y_{\pi/2}$$
$$B = Y_{\pi/2} - Y_{3\pi/2}$$

A corresponding orthogonal set of functions spanning the same space is found by the Gramm-Schmidt process to be

$$F_1,\ F_2,$$
$$F_3 = (6/\pi)^{1/2}A(\sin\theta\cos\phi - \tfrac{1}{2})$$
$$F_4 = (6/\pi)^{1/2}B(\sin\theta\cos\phi + \tfrac{1}{2}).$$

The flux may now be expanded as

$$f(r, \theta, \phi) = \sum_{i=1}^{4} f_i(r) F_i(\theta, \phi)$$

Substituting this into the differential equation gives

$$\begin{aligned}
&\sin\theta \cos\phi \sum_{i=1}^{N} F_i \frac{\partial}{\partial r} f_i \\
&\quad - \frac{\sin\theta \sin\phi}{r} \Big[(2\pi)^{-1/2} f_1 (\delta_{-\pi/2} - \delta_{\pi/2}) \\
&\quad + (2\pi)^{-1/2} f_2 (\delta_{\pi/2} - \delta_{3\pi/2}) \\
&\quad - (6/\pi)^{1/2} f_3 \sin\theta \sin\phi A + (6/\pi)^{1/2} f_3 (\sin\theta\cos\phi - 1/2) \\
&\qquad\qquad \cdot (\delta_{-\pi/2} - \delta_{\pi/2}) \\
&\quad - (6/\pi)^{1/2} f_4 \sin\theta \sin\phi B + (6/\pi)^{1/2} f_4 (\sin\theta\cos\phi + 1/2) \\
&\qquad\qquad \cdot (\delta_{\pi/2} - \delta_{3\pi/2}) \Big] \\
&\quad + \Sigma \sum_{i=1}^{4} f_i F_i = \frac{1}{2} \Sigma_S (2\pi)^{-1/2} (f_1 + f_2) + S(r, \theta, \phi)
\end{aligned}$$

To find the homogeneous moment equations, one sets the source term $S = 0$, multiplies by a basis function, F_i, and integrates over θ and ϕ. To carry out these operations the product $Y \cdot \delta$ must be defined.

H. Koenig has constructed a product space of generalized distributions and shown that this product space of distributions can be mapped into the space of distributions giving a product which contains a finite set of arbitrary constants.

In the case of $Y \cdot \delta$ the result is what one would expect intuitively, viz., $Y \cdot \delta = c\delta$ where c is an arbitrary constant (*4*). If one puts $Y_{\pi/2}\delta_{\pi/2} = c_1 \delta_{\pi/2}$ and $Y_{-\pi/2}\delta_{-\pi/2} = c_2 \delta_{-\pi/2}$, it is easy to show that one obtains the continuity equation only if $c_1 + c_2 = 1$. Also, the symmetry about the axis of the cylinder requires that $c_1 = c_2$. The same result can be obtained in other ways. For example, the rectangular functions A and B can be replaced by smoothed out functions by replacing Y and δ by the convolution of Y or δ with a "regularizing function" which may be chosen as a normalized Gaussian.

The resulting differential equations in the double P_1 approximation are

$$\begin{aligned}
h'_0 + g'_1 + (P/r)h_0 + (P/r)g_1 + 2\Sigma_a g_0 &= 0 \\
g'_0 + h'_1 + (4P/r)h_1 + 2\Sigma h_0 &= 0 \\
g'_0 + 3h'_1 - (3P/r)h_1 + 6\Sigma g_1 &= 0 \\
h'_0 + 3g'_1 - (3P/r)h_0 + (6P/r)g_1 + 6\Sigma h_1 &= 0
\end{aligned}$$

where

$$g_0 = f_1 + f_2, \qquad h_0 = f_1 - f_2$$

$$g_1 = (1/\sqrt{3})(f_3 + f_4), \qquad h_1 = (1/\sqrt{3})(f_3 - f_4)$$

and $P = 0$ for slabs, $P = 1$ for cylinders, and $P = 2$ for spheres.

The above equations in the sphere case agree with those given by A. Sauer (*5*). Sauer gives no argument for his definition of the product $Y \cdot \delta$ which agrees with ours.

The scalar flux is $\sqrt{2\pi} g_0$ and the current J is

$$J = \sqrt{2\pi}/2 (h_0 + g_1)$$

Putting $P = 1$ and $h_1 = g_1 = 0$ in the first two equations gives the double P_0 equations for cylinders. These are solved to yield

$$g_0 = aI_0(2\sqrt{\Sigma\Sigma_a} r) + bK_0(2\sqrt{\Sigma\Sigma_a} r)$$

$$h_0 = -(1/2\Sigma)[aI_1(2\sqrt{\Sigma\Sigma_a} r) + bK_1(2\sqrt{\Sigma\Sigma_a} r)]$$

One notes that $2\sqrt{\Sigma\Sigma_a} r$ is the same argument as occurs in the slab double P_0.

The behavior of the differential equations indicates that there are enough general solutions to construct both interior and exterior solutions. The general analytic solution is difficult to obtain except for the nonabsorption case where half-order Bessel functions and their integrals occur. For the cylinder the double P_1 zero absorption solution is

$$\begin{aligned}
g_0 = &-\frac{3a}{2} \int^r G_+(\alpha r')\, dr' - \frac{3b}{2} \int^r G_-(\alpha r')\, dr' + d \\
&+ c \Big[-\frac{3}{2} \int^r G_+(\alpha r') M_-(\alpha r')\, dr' \\
&\qquad + \frac{3}{2} \int^r G_-(\alpha r') M_+(\alpha r')\, dr' - \frac{2 l n r}{\pi \alpha} \Big]
\end{aligned}$$

$$\begin{aligned}
g_1 = &-\frac{a\alpha}{4\Sigma} I_{+9/2}(\alpha r) - \frac{b\alpha}{4\Sigma} I_{-9/2}(\alpha r) \\
&+ c \Big[-\frac{\alpha}{4\Sigma} I_{+9/2}(\alpha r) M_-(\alpha r) \\
&\qquad + \frac{\alpha}{4\Sigma} I_{-9/2}(\alpha r) M_+(\alpha r) + \frac{1}{3\Sigma\pi\alpha r} \Big]
\end{aligned}$$

$$\begin{aligned}
h_0 = &\frac{a\alpha}{4\Sigma} I_{+9/2}(\alpha r) + \frac{b\alpha}{4\Sigma} I_{-9/2}(\alpha r) \\
&+ c \Big[\frac{\alpha}{4\Sigma} I_{+9/2}(\alpha r) M_-(\alpha r) - \frac{\alpha}{4\Sigma} I_{-9/2}(\alpha r) M_+(\alpha r) + \frac{1}{\Sigma\pi\alpha r} \Big]
\end{aligned}$$

$$\begin{aligned}
h_1 = &aI_{+7/2}(\alpha r) + bI_{-7/2}(\alpha r) + c[I_{+7/2}(\alpha r) M_-(\alpha r) \\
&\qquad - I_{-7/2}(\alpha r) M_+(\alpha r)]
\end{aligned}$$

where

$$G_\pm(\alpha r) = (5/r) I_{\pm 7/2}(\alpha r) + \alpha I_{\pm 9/2}(\alpha r)$$

$$M_\pm(\alpha r) = \int^r [I_{\pm 7/2}(\alpha r')/r']\, dr'$$

$$\alpha^2 = 12\Sigma^2$$

Again the argument α is the same as for the nonabsorption double P_1 slab case.

For the spherical case, somewhat similar expressions involving Bessel functions of order fifteen halves can be obtained.

The double P_1 for cylinders and spheres does not seem suitable for hand calculations. However, on physical grounds the methods should have features similar to the double P for slabs and thus should be useful for the same type of problems (*6*).

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The Age of U²³³ Fission Neutrons in Water

The age to indium resonance energy (1.4 ev) of U^{233} fission neutrons slowing down in water has been calculated with the SLAG code (*1*) on the University of Michigan IBM-650. The fission spectrum used was measured at Oak Ridge (*2*) with the low energy portion ($E < 1.3$ Mev) taken to be the (U^{235}) Watt spectrum (*3*). This spectrum, and the Watt spectrum, both normalized to unit area, are given in Fig. I as a function of $u = \ln(10^7/E)$ where E is in electron-volts. The crosses indicate the experimental points, and the smooth curve has been drawn by eye.

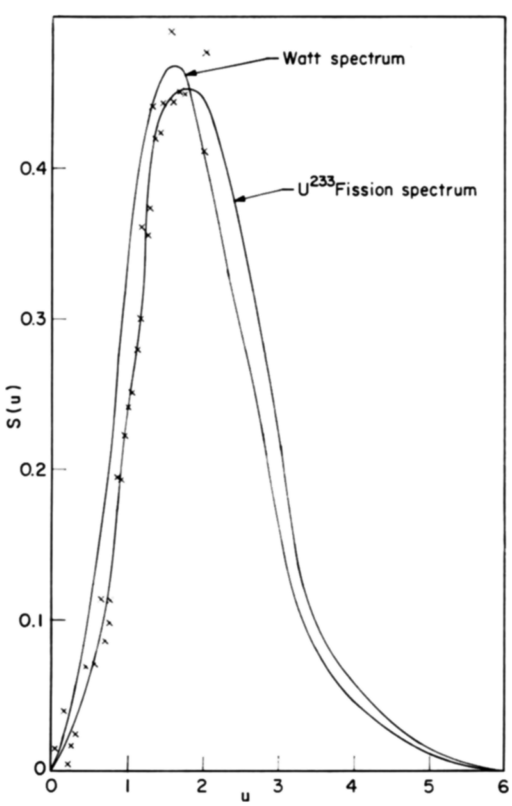

Fig. I. Fission neutron spectrum of U^{233} and U^{235}. The U^{235} spectrum is the Watt spectrum given by $\chi\ (E) = 0.484\ e^{-E} \sinh \sqrt{2E}$.

The result for τ (1.4 ev) is 23.0 ± 3 cm² compared to a value of 25.3 cm² from the Watt spectrum. The error limits were obtained from calculations for curves passing through the maximum and minimum error points of the measured spectrum. In order to test the sensitivity to the fit to the experimental points, a calculation was made for a curve generated by connecting adjacent experimental points, with no change being observed in the age. Finally, a calculation was made for the measured fission spectrum of U^{235} reported in reference *1*, which differs slightly from the Watt spectrum, in order to test the possibility of consistent errors in the fission spectrum measurements. However, this spectrum led to a value of τ identical with that given by the Watt spectrum.

The results of these calculations indicate that the age of U^{233} fission neutrons is probably about 8 or 9 per cent lower than the age of U^{235} fission neutrons, a fact which has important implications in the measurement of η of U^{233} by critical experimental techniques, such as are now being used at ORNL (*4*). In addition, it indicates that the losses to fast leakage from a U^{233} system will be somewhat less than those from a U^{235} system, which improves the possibility of breeding. It is clear, however, that an age experiment with U^{233} fission neutrons must be performed because of the fairly large error limits on the measured spectrum which lead to the error limits of $\pm$ 3 cm² in the age.

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Note on Position-Dependent Spectra

Near boundaries between dissimilar media (A, B, etc.), the spatial behavior of the thermal neutron flux is predicted incorrectly by normal diffusion theory, in which energy averages of the constants are taken over the asymptotic spectra of A, B, etc. To some extent transport corrections are necessary but the predominant effect is that due to a continuous change of spectrum with position in going from one medium to another.

In particular, the observed peaking from a slab water gap immersed in a multiplying medium is always higher than that predicted by normal diffusion theory. On the other hand, the first correction term arising from a one-