$$V_2 = \begin{pmatrix} S - RC \\ RS/2 + \sqrt{T^2 - 1} \end{pmatrix} \tag{6}$$

where we have abbreviated

$$S = \sin \mu\delta; \qquad C = \cos \mu\delta \tag{7}$$

$$T = C + (RS/2) \tag{8}$$

with eigenvalues

$$\lambda_{\substack{1\\2}} = T \pm \sqrt{T^2 - 1}. \tag{9}$$

That being the case, we have

$$QP = P\Lambda \tag{10}$$

where

$$P = \begin{pmatrix} S - RC & S - RC \\ RS/2 - \sqrt{T^2 - 1} & RS/2 + \sqrt{T^2 - 1} \end{pmatrix} \tag{11}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{12}$$

or

$$Q = P\Lambda P^{-1}. \tag{13}$$

Thus,

$$Q^N = P\Lambda^N P^{-1} \tag{14}$$

or, more explicitly,

$$Q^N = \frac{1}{\Delta} \begin{pmatrix} S - RC & S - RC \\ \frac{RS}{2} - \sqrt{T^2 - 1} & \frac{RS}{2} + \sqrt{T^2 - 1} \end{pmatrix} \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \begin{pmatrix} \frac{RS}{2} + \sqrt{T^2 - 1} & CR - S \\ \sqrt{T^2 - 1} - \frac{RS}{2} & S - RC \end{pmatrix} \tag{15}$$

where

$$\Delta = 2\sqrt{T^2 - 1}\,(S - RC). \tag{16}$$

By direct calculation from Eqs. (3) and (15), we find the surprisingly simple critical equation

$$(S, C)Q^N \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{S}{2\sqrt{T^2 - 1}} \{\lambda_1^{N+1} - \lambda_2^{N+1}\} = 0. \tag{17}$$

By inspection, one can verify that the only admissible solutions of Eq. (17) are those for which

$$(\lambda_1/\lambda_2)^{N+1} = 1. \tag{18}$$

If we write $T = \cos \psi$ in Eq. (9), and substitute in Eq. (18)

$$e^{2i(N+1)\psi} = 1 \tag{19}$$

that is, the critical values of $\psi = \cos^{-1} T$ are

$$\psi_j = j\pi/(N + 1) \quad (j = 1, 2, \cdots, 2N + 1) \tag{20}$$

and the critical equation, from Eqs. (8) and (20), takes the simple form

$$\cos \mu\delta + \frac{R}{2} \sin \mu\delta = \cos \frac{j\pi}{N + 1} \qquad (j = 1, 2, \cdots, 2N + 1) \tag{21}$$

Equation (21) is understood to be solved for each j, and the smallest positive root for k so obtained is the desired eigenvalue.

It is of interest to note that Eq. (21) can be solved explicitly for the critical spacing δ, in the form

$$\delta_{\text{crit}} = \frac{1}{\mu} \cos^{-1} \left\{ \frac{\cos [j\pi/(N + 1)] \pm \frac{1}{2}R\sqrt{\sin^2[j\pi/(N + 1)] + R^2/4}}{1 + R^2/4} \right\} \tag{22}$$

which is to be interpreted in a manner similar to Eq. (21).

REFERENCE

1. B. Wolfe, "General Theory of Control Sheets," *Nuclear Sci. and Eng.* **4**, 635-648 (1958).

H. S. Wilf

Nuclear Development Corporation of America
White Plains, New York
Received November 24, 1958

A Simple Treatment for Effective Resonance Absorption Cross Sections in Dense Lattices

It has recently been shown by Chernick *et al.* (*1, 2*) that effective resonance absorption cross sections can be computed with the same expressions for both homogeneous mixtures of absorber and moderator and also for isolated[1] lumps of absorber in moderator. This result was obtained by making for the isolated lump case, the so-called Wigner or canonical approximation to the neutron escape probability from a lump. Let S denote lump area, V_0 lump volume, V_1 moderator volume per lump, Σ_0 macroscopic cross section in lump, and Σ_1 moderator cross section. In this notation, it was found that the quantity $S/4V_0 = s_0$ plays the same role for the heterogeneous case that the moderator cross section per absorber atom $(\Sigma_1 V_1/V_0)$ plays in the homogeneous case. The quantity s_0 was interpreted as a pseudo-cross section representing escape from the lump (*2*).

For the case of dense lattices with closely spaced lumps, it has been customary to apply Dancoff corrections (*3*) to the isolated lump case. This is frequently a quite complicated procedure. It is the purpose of this note to indicate how the canonical treatment may be generalized to the case of closely spaced lumps and to obtain a transition between the isolated lump and homogeneous cases. The result of such a generalization is very simple; namely, in general the quantity s_0 is to be replaced (in all isolated lump expressions) by τ_0, where

$$\tau_0 = \frac{s_0 \Sigma_1}{\Sigma_1 + s_0 (V_0/V_1)}. \tag{1}$$

In the following, we shall first give a heuristic justification of this recipe and then note some of its desirable properties.

We assume, as usual, that neutrons arrive at any energy E uniformly in space within either the absorber lump

[1] By isolated we mean that separation between lumps is large compared to a moderator mean free path.

(subscript 0) or moderator (subscript 1). We wish then to compute the probabilities, P_i, that a neutron originating uniformly in region i makes its next collision in the same region (although possibly after traversing the other region). Let P_i^0 be the corresponding probabilities for isolated lumps, and let G_i^j be the probability that a neutron incident on region i, after j previous traversals of region $i (j \geqq 0)$, makes a collision in region i before leaving. For simplicity, we now assume that G_i^j is independent of j and drop the superscript j, but such an assumption is not really necessary at this stage. We then find

$$1 - P_0 = (1 - P_0^0)[G_1 + (1 - G_1)(1 - G_0)G_1 + \cdots (1 - G_1)^n(1 - G_0)^n G_1 + \cdots]$$

or

$$1 - P_0 = (1 - P_0^0) \frac{G_1}{1 - (1 - G_1)(1 - G_0)}. \quad (2)$$

The Wigner or canonical approximation consists in setting, as in reference *1*

$$1 - P_0^0 = s_0/(\Sigma_0 + s_0) \quad (3a)$$

$$1 - G_0 = s_0/(\Sigma_0 + s_0) \quad (3b)$$

$$1 - G_1 = s_1/(\Sigma_1 + s_1) \quad (3c)$$

with $s_1 = S/4V_1 = s_0 V_0/V_1$. Substituting Eqs. (3) in Eq. (2), we find that

$$1 - P_0 = \tau_0/(\Sigma_0 + \tau_0) . \quad (4)$$

By comparing Eqs. (3a) and (4) we see that, in this general case, τ_0 has replaced s_0 of the isolated lump case. A similar result may be derived for P_1

$$1 - P_1 = \tau_1/(\Sigma_1 + \tau_1) \qquad \tau_1 = s_1\Sigma_0/(\Sigma_0 + s_0). \quad (5)$$

Since all effective cross sections may be derived[2] in terms of P_0 and P_1, it follows that general effective cross sections will be obtained from isolated lump expressions by replacing s_0 by τ_0.

In addition to its simplicity, this canonical recipe has a number of desirable properties:

(a) It gives the isolated lump limit when the lumps are widely spaced. This limit is obtained as $s_1/\Sigma_1 \to 0$ (s_i is a reciprocal mean chord length in region i), for which we see, from Eq. (1), that $\tau_0 \to s_0$ and we have the isolated lump case.

(b) It gives the homogeneous mixture limit when lump and moderator regions are thin, i.e., when $\Sigma_1/s_1 \ll 1$ and $\Sigma_0/s_0 \ll 1$. In this case we see from Eqs. (1), (4), and (5) that

$$P_0 \simeq \frac{\Sigma_0}{\Sigma_0 + \Sigma_1 (V_1/V_0)} \simeq 1 - P_1 \quad (6)$$

which means that the probability of a neutron colliding with an absorber nucleus is simply equal to absorber cross section divided by total cross section, and is independent of where the neutron originated. Clearly this is just the homogeneous case.

(c) Our expressions satisfy the exact reciprocity relation

$$(1 - P_0)\Sigma_0 V_0 = (1 - P_1)\Sigma_1 V_1 \quad (7)$$

as is readily verified by substitution and a little algebra.

Calculations in which s_0 is replaced by τ_0 have been made for some practical dense lattices. It is concluded that the use of τ_0 for dense lattices is about as accurate as is the use of s_0 for isolated lumps. Thus we have not lost appreciable accuracy in generalizing the isolated canonical approximation.

It is straightforward to generalize the above results further to allow for the presence of moderator in the absorbing lumps or to treat cells which have a central cluster of absorber lumps plus moderator. The interpretation of τ_0 as a generalized leakage pseudo-cross section may be maintained throughout.

[2] In fact they may be derived from P_0 alone if Eq. (7) is employed to eliminate P_1.

REFERENCES

1. J. Chernick and R. Vernon, Some Refinements in the Calculation of Resonance Integrals. *Nuclear Sci. and Eng.* **4**, 649 (1958).

2. B. I. Spinrad, J. Chernick, and N. Corngold, *Proc. Second Intern. Conf. Peaceful Uses Atomic Energy*, Paper P/1847 (1958).

3. S. M. Dancoff and M. Ginsburg, CP-2157 (1944).

George I. Bell

Los Alamos Scientific Laboratory
Los Alamos, New Mexico
Received November 24, 1958