NUCLEAR SCIENCE AND ENGINEERING: **4**, 495–498 (1958)

LETTERS TO THE EDITORS

Solution of the Time-Dependent Thermal Neutron Diffusion Equation[1]

This note is concerned with solution of the time-dependent thermal neutron diffusion equation (with extraneous source term) by means of the Laplace transform; this method of solution does not assume that the spatial distribution of neutrons for a reactor is described by the fundamental eigenfunction. The limitations and implications of this latter assumption have been discussed in a Geneva Conference paper by L. N. Usachoff (*1*). The Laplace transform method is advantageous in that it may be used to study time-dependent reactor problems in which the spatial distribution of flux cannot be approximated by the fundamental eigenfunction; such problems arise in startup of a reactor and situations in which the reactor is pulsed by an extraneous source of neutrons. Therefore, our results constitute a generalization of the results of elementary reactor kinetics theory.

To be specific, let us develop our results for a bare homogeneous sphere[2] (of extrapolated radius $\bar{a}$) of multiplying material, and adopt a Fermi age theory for our slowing-down model and neglect delayed neutron effects. In this case, the neutron diffusion equation (*2*) is given by

$$L^2\nabla^2\phi(r, t) + [k \exp(-B^2\tau) - 1]\,\phi(r, t) + [S(r, t)/\Sigma_a] = l_0 \frac{\partial\phi(r, t)}{\partial t}, \qquad (t > 0) \quad (1)$$

where $\phi(r, t)$ = thermal neutron flux, $S(r, t)$ = extraneous source term, L = thermal neutron diffusion length, k = multiplication constant, B = buckling, τ = Fermi age, Σ_a = macroscopic absorption cross section for thermal neutrons, l_0 = infinite medium mean lifetime of thermal neutrons.

Let us now take the Laplace transform[3] (with respect to time) of Eq. (1), which yields

$$L^2\nabla^2\bar{\phi}(r, p) + [k \exp(-B^2\tau) - 1]\bar{\phi}(r, p) + \bar{S}(r, p) = l_0[p\bar{\phi}(r, p) - \phi(r, 0)] \quad (2)$$

where

$$\bar{\phi}(r, p) = \int_0^\infty \exp(-pt)\phi(r, t)\,dt$$
$$\bar{S}(r, p) = (1/\Sigma_a) \int_0^\infty \exp(-pt)S(r, t)\,dt. \quad (3)$$

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[1] This work was performed under the auspices of the U. S. Atomic Energy Commission.

[2] Parallel results may be obtained for an infinite slab geometry.

[3] Some of the elementary properties of the Laplace transform are assumed without elaboration. For further details the reader is referred to the book of Churchill (*3*).

We may rewrite Eq. (2) in the form

$$\nabla^2\bar{\phi}(r, p) - A^2\bar{\phi}(r, p) = F(r, p) \tag{4}$$

where

$$A^2 = (l_0/L^2)\{p - (1/l_0)\,[k \exp(-B^2\tau) - 1]\}$$

$$F(r, p) = -(1/L^2)[S(r, p) + l_0\phi(r, 0)].$$

The solution (with spherical symmetry) to Eq. (4) may be written in the form

$$\bar{\phi}(r, p) = C[\sinh(Ar)/r] + \bar{G}(r, p) \tag{5}$$

where $\bar{G}(r, p)$ is a particular solution corresponding to the inhomogeneous part of Eq. (4). In the particular case of a point source, Eq. (5) must be modified; this will be illustrated by our example.

From the original boundary condition

$$\phi(\bar{a}, t) = 0 \tag{6}$$

we may show that

$$\bar{\phi}(\bar{a}, p) = 0, \tag{7}$$

and hence, applying condition (7), Eq. (5) becomes

$$\bar{\phi}(r, p) = \bar{G}(r, p) - (\bar{a}/r)\bar{G}(\bar{a}, p)[\sinh(Ar)/\sinh(A\bar{a})]. \tag{8}$$

We now propose to find the inverse transform of Eq. (8); this may be done by means of the *complex inversion integral* [see p. 148 of reference (*3*)] or by making use of tables of Laplace transforms. We shall make use of the latter technique. In order to facilitate finding the inverse transform of (8), we make the substitution

$$p' = p - (1/l_0)[k \exp(-B^2\tau) - 1] \tag{9}$$

and thus obtain a new function of p' (denoted by a subscript)

$$\phi_1(r, p') = G_1(r, p') - (\bar{a}/r)G_1(\bar{a}, p')[\sinh(r\sqrt{l_0p'}/L)/\sinh(\bar{a}\sqrt{l_0p'}/L)]. \tag{10}$$

In the object plane, the original function and the inverse function resulting from (10) are related by

$$\phi(r, t) \exp(-1/l_0)[k \exp(-B^2\tau) - 1]t = \phi_1(r, t). \tag{11}$$

The form of Eq. (11) is similar to that assumed by A. F. Henry (*4*) in his work on reactor kinetics. We may express the inverse transform of (10) in the form of convolution integral (4.1-20) and transform (4.25-9) of the Bateman Project Tables (*5*). This gives for the function $\phi_1(r, t)$

$$\phi_1(r, t) = G_1(r, t) - (L^2/rl_0)\int_0^t G_1(a, t - t')\,\frac{\partial}{\partial r}\,\theta_4[(r/2\bar{a}) \mid (\pi L^2 it'/\bar{a}^2 l_0)]\,dt', \tag{12}$$

$$(-a < r < a)$$

where θ_4 is the *fourth Jacobi theta function* (*6*); the expression (12) furnishes an explicit solution to our problem for a general extraneous source term. The theta functions have

attractive function theoretic properties in that they are entire functions and can be represented by series whose convergence is extremely rapid,

$$\theta_4 [(r/2\bar{a}) \mid (\pi L^2 i t'/\bar{a}^2 l_0)] = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp (-\pi^2 n^2 L^2 t'/\bar{a}^2 l_0) \cos (n\pi r/\bar{a}. \quad (13)$$

This latter fact is important from the standpoint of numerical analysis. In case the integral (12) is not capable of being evaluated explicitly, it will be necessary to resort to numerical quadrature. In this conjunction, there has been some tabulation of the derivatives of the theta functions (7). Since θ_4 is an entire function of r, we may differentiate series (13) termwise to arrive at an expansion for the derivative $\partial\theta_4/\partial r$; this series may be used as a basis of calculation of the integral (12).

As an example of the technique discussed above, let us assume that we wish to examine the time behavior of a spherical reactor which has no initial flux, but which is activated by a pulsed central point source of S_0 neutrons/cm^3/sec at time $t = 0$; the source term for this problem is given by the expression

$$S(r, t) = S_0 \delta(r) \delta(t) \quad (14)$$

where $\delta(r)$ and $\delta(t)$ are the familiar Dirac delta functions. In this case, the Laplace transform of the source function (13) gives

$$\bar{S}(r, p) = S_0 \delta(r)/\Sigma_a \quad (15)$$

and to take account of the point source, we take the solution of Eq. (4) in the form

$$\bar{\phi}(r, p) = (1/r)[C_1 \exp (Ar) + C_2 \exp (-Ar)] \quad (16)$$

and then apply the method of satisfaction of conditions at the source outlined by Glasstone and Edlund [see p. 107 of reference (2)] to determine the additional arbitrary constant of Eq. (15). This finally gives an expression for the Laplace-transformed flux,

$$\phi(r, p) = (S_0/4\pi D r \Sigma_a)[\sinh A(\bar{a} - r)/\sinh A\bar{a}]. \quad (17)$$

The inverse transform $\phi_1(r, t)$ of Eq. (16) may be found directly in terms of the derivative of the fourth theta function, θ_4, using the same transform pair as was used in Eq. (12),

$$\phi_1(r, t) = (-S_0 L^2/4\pi D r \bar{a} l_0) \frac{\partial}{\partial_n} \phi_4[(\bar{a} - r)/2\bar{a} \mid (\pi L^2 i t'/\bar{a}^2 l_0)], \qquad (0 < r < 2\bar{a}). \quad (18)$$

Many more examples could be presented (e.g., the behavior of a reactor with a stationary flux and subjected to a change in reactivity). The practical limitations of any further development of theory along the lines of this paper are brought about by the inadequacies of the physical model from which Eq. (1) was evolved. For certain reactor situations, a more adequate neutron slowing-down model is furnished by multigroup diffusion theory; for this reason it would be valuable to develop a time-dependent multigroup theory. It will be virtually impossible to find the inverse Laplace transform for a multigroup problem as was done for the problem of this note because of the formidable mathematical reductions involved, and therefore, some numerical method for inversion of the Laplace transform must be used. At the present time, the author is investigating various methods for numerical inversion of the Laplace transform in conjunction with the study of time-dependent multigroup reactor problems.

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Received April 16, 1958

Critical Measurements on UO_3-H_3PO_4 Solutions[1]

During the course of homogeneous reactor fuel studies, critical measurements were carried out with solutions of UO_3 (93.5% U^{235}) dissolved in 4.3 molar H_3PO_4. A brief description of these measurements, their results, and an evaluation of some calculations is presented here.

The basic fuel geometry for critical mass determinations was a right circular cylinder, 15.75 cm in radius. Fuel solution was held in a stainless steel container (0.1 cm wall thickness) surrounded radially and axially by a 9 cm iron reflector. The experimental apparatus was located in the Pajarito Site remotely-controlled critical assembly facility at LASL.

Determinations of critical masses were made as follows. The container was filled to various heights with phosphoric acid. An appropriately canned neutron source was located near the center of the phosphoric acid. Unmultiplied counts were taken with BF_3 proportional counters positioned outside the iron reflector. Multiplied counts were made after replacing the phosphoric acid with fuel solution.

Filling the experimental assembly with fuel solution was done with the top reflector raised, and with the container (together with the bottom reflector) withdrawn below the radial reflector. The solution was transferred into the container from 5-inch diameter 20-liter storage cans. These were pressurized to produce a slow flow of fuel, through Tygon tubing, to the experimental setup. A weighing device holding the storage cans indicated completion of the addition of a preselected amount of fuel. The container and bottom reflector were raised into the radial reflector, and the top reflector was lowered inside the container to the level of the fuel solution, by means of a remotely controlled assembly mechanism, after all personnel had left the experimental area.

[1] Work performed under the auspices of the U. S. Atomic Energy Commission.