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LETTERS TO THE EDITORS

A Dynamic Programming Solution to a Problem in Heavy Water Production

In a recent paper (*1*), the problem of designing a feasible distillation plant for the production of heavy water was discussed. Since large amounts of steam are required, the cost of the process would ordinarily be prohibitive from the stand-point of plant size, and the quantity of fuel required for heating. The authors, however, have in mind the use of geothermal steam (*2*), which renders the essential constraint that of plant size.

Under various assumptions concerning the nature of the distillation process, the problem of determining the most efficient cascading process is reduced in the first paper cited above to a multi-dimensional maximization problem, which the authors solved approximately using an iterative technique. Since the actual physical process is multi-stage, it may be expected that the theory of dynamic programming (*3*, *4*) will furnish a more systematic computational solution to questions of this type, and to those arising from more realistic assumptions. In this letter, we shall consider only one case treated by the cited authors.

The Analytic Problem

Following the discussion of Marchetti (*5*), and that given in (*1*), the mathematical problem is that of minimizing the plant size for an m-cascade process, as given by

$$V = \sum_{i=1}^{m} \log \left(\frac{a_i - \rho_i}{1 - \rho_i} \right) \Big/ N_i \rho_i \tag{1}$$

subject to the constraints

$$\prod_{i=1}^{m} a_i = I, \qquad a_i \geq 1. \tag{2}$$

Here I is the desired total enrichment, and a_i is the enrichment per stage. The quantities ρ_i are known functions of the a_i determined by the relations

$$\frac{\rho_i(a_i - 1)}{(a_i - \rho_i)(1 - \rho_i)} = \log \left(\frac{1 - \rho_i}{a_i - \rho_i} \right) \tag{3}$$

with $0 < \rho_i < 1$ (*5*). Finally,

$$\begin{aligned} N_1 &= 1.5 \times 10^{-4}, N_2 = a_1 N_1, \cdots, N_{i+1} = a_i N_i, \; i = 1, 2, \cdots, m - 1, \\ N_{m+1} &= I N_1 . \end{aligned} \tag{4}$$

In the case treated in (*1*), $I = 300$.

Setting

$$g(a_i) = \frac{1}{\rho_i} \log \left(\frac{a_i - \rho_i}{1 - \rho_i} \right) \tag{5}$$

and referring to (4), we see that the problem is equivalent to that of minimizing the function

$$V = g(a_1) + \frac{g(a_2)}{a_1} + \frac{g(a_3)}{a_1 a_2} + \cdots + \frac{g(a_m)}{a_1 a_2 \cdots a_{m-1}} \tag{6}$$

over all a_i subject to the constraints

$$a_1 a_2 \cdots a_m = I, \qquad a_i \geq 1. \tag{7}$$

Variational problems of this type, which can be quite difficult to treat by conventional methods, can be resolved in a simple fashion computationally, and occasionally analytically, using the techniques of dynamic programming.

Dynamic Programming Formulation

Let us introduce the sequence of functions $\{f_k(x)\}$, defined as follows:

$$\left[f_k(x) = \underset{\{a_i\}}{\mathrm{Min}} \left[g(a_1) + \frac{g(a_2)}{a_1} + \frac{g(a_3)}{a_1 a_2} + \cdots + \frac{g(a_k)}{a_1 a_2 \cdots a_{k-1}} \right] \right. \tag{8}$$

for $k = 2, 3, \cdots$, where the a_i are subject to

$$a_1 a_2 \cdots a_k = x, \; a_i \geq 1 \tag{9}$$

and x may assume any positive value, greater than one.

The function $f_2(x)$ is readily determined from (8) and (9), namely

$$f_2(x) = \underset{a_1, a_2}{\mathrm{Min}} \, [g(a_1) + g(a_2)/a_1] \tag{10}$$

where $a_1 a_2 = x$, a_1 , $a_2 \geq 1$.

Let us now derive a recurrence relation connecting $f_{k+1}(x)$ with $f_k(x)$. Writing

$$f_{k+1}(x) = \underset{\{a_i\}}{\mathrm{Min}} \left[g(a_1) + \frac{1}{a_1} \left\{ g(a_2) + \frac{g(a_3)}{a_2} + \cdots + \frac{g(a_{k+1})}{a_2 a_3 \cdots a_k} \right\} \right] \tag{11}$$

we see that

$$f_{k+1}(x) = \underset{a_1 \geq 1}{\mathrm{Min}} \left[g(a_1) + \frac{1}{a_1} f_k \left(\frac{x}{a_1} \right) \right] \tag{12}$$

for $k = 2, 3, \cdots$.

This is an application of the principle of optimality (*3, 4*), which in this case has the following simple physical interpretation: "Whatever enrichment is attained in the first cascade, at whatever cost in volume of plant, the remaining enrichment is to be obtained using minimum plant volume."

The solution of the original minimization is thus reduced to determining the sequence $\{f_k(x)\}$, using the recurrence relation in (12). This is a very simple process which can be carried out via a direct hand computation, a direct computation on a digital computer, or by using variational techniques.

Discussion

The technique discussed in the preceding section enables us to consider more realistic processes. If we allow inhomogeneous cascades, we are confronted by the problem of minimizing an expression of the form

$$g_1(a_1) + \sum_{i=2}^{m} \frac{g_i(a_i)}{a_{i-1}a_i \cdots a_{m-1}} \tag{13}$$

where the sequence $\{g_i(x)\}$ is known, over the same region as above.

To treat a problem of this type introduce the sequence of functions

$$f_k(x) = \underset{\{a_i\}}{\text{Min}} \left[g_k(a_k) + \sum_{i=k+1}^{m} \frac{g_i(a_i)}{a_{i-1}a_i \cdots a_{m-1}} \right] \tag{14}$$

where the a_i are subject to

$$a_k a_{k+1} \cdots a_m = x, \qquad a_i \geq 1 \tag{15}$$

for $k = 1, 2, \cdots, m - 1$.

Then

$$f_{m-1}(x) = \underset{a_{m-1},a_m}{\text{Min}} \left[g_{m-1}(a_{m-1}) + \frac{g_m(a_m)}{a_{m-1}} \right] \tag{16}$$

over $a_{m-1}a_m = x$, a_{m-1} , $a_m \geq 1$, and as before,

$$f_k(x) = \underset{a_k \geq 1}{\text{Min}} \left[g_k(a_k) + \frac{1}{a_k} f_{k+1}\left(\frac{x}{a_k}\right) \right]. \tag{17}$$

The computational solution is similar to that for Eq. (12).

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Critical Equation for the Bare Water-Moderated Reactor

A critical equation for the bare water-moderated reactor has been derived using the Goertzel-Selengut method. Neutron slowing down was assumed to be due to hydrogen alone, and the fission source was taken to be monoenergetic.

The flux at lethargy u in the slowing down region satisfies Eq. (1)

$$D(u)\nabla^2\phi(r, u) - [\Sigma_a(u) + \Sigma_{SH}(u)]\phi(r, u) + q(r, u) = 0, \tag{1}$$