For the three-group method, the parameters given by Eqs. (C1), (C2), and (C3) were calculated for three broad energy groups. Group 1 covers energy range 10 MeV to 111 keV, g = 1 to 18. Group 2 covers energy range 111 keV to 5.53 keV, g = 10 to 30. Group 3 covers energy range 5.53 keV to 47 eV, g = 31 to 50.

ACKNOWLEDGMENT

This work was performed under the auspices of the U.S. Atomic Energy Commission.

Synthesis Solutions of the Multigroup Boltzmann Equation for Slabs

W. Schwetje

Institut für Neutronenphysik und Reaktortechnik
Kernforschungszentrum Karlsruhe, Germany

Received December 21, 1970
Revised May 24, 1971

I. INTRODUCTION

In recent publications[1-7] the methods of flux synthesis have been applied to the approximate solution of the linear Boltzmann equation for neutron transport. Practical results have been presented for several monoenergetic slab problems[1-6] and for multigroup critical slab problems.[7] A survey of the methods developed in the last few years is given in Ref. 8. This paper presents some methods for the solution of the multigroup Boltzmann equation for slab geometry with inhomogeneous boundary conditions.[9] The methods are based on half-range expansions of the unknown solution in known functions of the angle variable, in known functions of the angle and the energy variable, in known functions of the space variable, or in known functions of the space and the energy variable. The unknown functions of these expansions are determined by the "method of weighted residuals,"[10-12] which yields reduced multigroup problems when the unknown solution is expanded in functions of the angle or the space variable, and reduced one-group problems when the solution is expanded in functions of the energy variable also. The reduced equations are systems of linear first-order differential equations or systems of Fredholm integral equations of the second kind with degenerate kernel which are solved numerically.

II. FLUX SYNTHESIS

Assuming isotropic scattering in the LS and treating the energy dependence in the multigroup model,[13] the Boltzmann equation for slab geometry without independent volume sources is given by Eq. (1)

$$\mu \frac{\partial}{\partial x} f^g(x,\mu) + \sigma_t^g(x) f^g(x,\mu) = \frac{1}{2} \sum_{j=1}^{G} \sigma_s^{j\to g} \int_{-1}^{1} f^j(x,\mu')\, d\mu' \qquad g = 1, 2, \ldots, G \quad . \tag{1}$$

The boundary conditions, Eqs. (2), describe the incident neutron fluxes on the left-hand and right-hand sides of the slab, respectively.

$$f^g(0,\mu) = r_l^g(\mu) \qquad 0 < \mu \le 1 \tag{2a}$$

$$f^g(d,\mu) = r_r^g(\mu) \qquad -1 \le \mu < 0 \qquad g = 1, 2, \ldots, G \; . \tag{2b}$$

Below we always assume that

$$r_r^g(\mu) \equiv 0 \quad , \tag{3}$$

that is, we only treat problems with incident neutrons on the left-hand side and homogeneous boundary conditions on the right-hand side. The notation is

d = slab thickness

x = spatial variable

μ = cos θ

θ = angle between x axis and direction of the neutrons

σ_t^g = total macroscopic cross section in group g

$\sigma_s^{j\to g} = \sigma_{sc}^{j\to g} + \nu \chi^g \sigma_f^j$

$\sigma_{sc}^{j\to g}$ = macroscopic cross section for scattering from group j to group g

ν = mean number of fission neutrons

χ^g = fission spectrum

σ_f^j = macroscopic fission cross section in group j

A. *Boundary Conditions*

Setting

$$f^g(x,\mu) = f^{g0}(x,\mu) + f^{g1}(x,\mu) \quad , \tag{4}$$

where $f^{g0}(x,\mu)$ describes the uncollided neutron flux in group g, we obtain Eq. (5) for $f^{g0}(x,\mu)$.

$$\mu \frac{\partial}{\partial x} f^{g0}(x,\mu) + \sigma_t^g(x) f^{g0}(x,\mu) = 0 \qquad g = 1, 2, \ldots, G \quad . \tag{5}$$

The uncollided flux $f^{g0}(x,\mu)$ satisfies the inhomogeneous boundary conditions, Eqs. (2), whereas the scattered flux $f^{g1}(x,\mu)$, which includes the fission neutrons, satisfies homogeneous boundary conditions. The solution of Eq. (5), taking into account the boundary conditions Eqs. (2) and (3), is given by Eq. (6)

$$f^{g0}(x,\mu) = \begin{cases} r_l^g(\mu) \exp\left[-\int_0^x \sigma_t^g(x')\, dx' \cdot \mu^{-1}\right] & 0 < \mu \le 1 \\ 0 & -1 \le \mu \le 0 \end{cases} \quad . \tag{6}$$

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For the scattered flux $f^{g1}(x,\mu)$ we obtain the inhomogeneous Eq. (7) together with homogeneous boundary conditions

$$\mu \frac{\partial}{\partial x} f^{g1}(x,\mu) + \sigma_t^g(x) f^{g1}(x,\mu) = \frac{1}{2} \sum_{j=1}^{G} \sigma_s^{j\to g} \int_{-1}^{1} f^{j1}(x,\mu')\, d\mu' + \frac{1}{2} \sum_{j=1}^{G} \sigma_s^{j\to g} \int_{-1}^{1} f^{j0}(x,\mu')\, d\mu' \qquad g = 1, 2, \ldots, G \quad . \quad (7)$$

If we now try to find approximate solutions for Eq. (1) which exactly satisfy the boundary conditions Eqs. (2) and (3), it is sufficient to find approximate solutions for Eq. (7) which exactly satisfy homogeneous boundary conditions. Another possibility for exactly satisfying the boundary conditions Eqs. (2) and (3) is pointed out later in this paper.

B. Expansion in Functions of the Angle Variable

The unknown solution of Eq. (7) is expanded as follows [Eqs. (8)]

$$f^{g1}(x,\mu) = \sum_{i=1}^{N} [\phi_i^{g+}(x)\psi_i^{+}(\mu) + \phi_i^{g-}(x)\psi_i^{-}(\mu)] \qquad g = 1, 2, \ldots, G \quad (8a)$$

$$\psi_i^{+}(\mu) = 0 \qquad 0 < \mu \le 1 \qquad (8b)$$

$$\psi_i^{-}(\mu) = 0 \qquad -1 \le \mu \le 0 \qquad i = 1, 2, \ldots, N \quad . \quad (8c)$$

The linearly independent expansion functions or modes $\psi_i^{\pm}(\mu)$ are to be chosen by the user so that they are well suited to the problem to be solved. The coefficient functions $\phi_i^{g\pm}(x)$ are determined by the method of weighted residuals[10-12]: Substituting Eq. (8) into Eq. (7), multiplying the resulting equation in turn with $2N$ linearly independent weighting functions $w_j^{\pm}(\mu)$, and integrating over μ, we obtain, after some algebra, Eq. (9) from which the coefficient functions $\phi_i^{g\pm}(x)$ can be calculated. The weighting functions $w_j^{\pm}(\mu)$ are half-range functions in the same sense as $\psi_j^{\pm}(\mu)$, Eqs. (8b) and (8c), and are also chosen by the user. These weighting functions determine in which sense the approximate solution Eq. (8a) will become a "best" approximation.[11]

$$\frac{d}{dx} \tilde{\phi}^g(x) + A^g(x)\tilde{\phi}^g(x) = B(x) \sum_{j<g} \sigma_{sc}^{j\to g}(x)\tilde{\phi}^j(x) + \nu\chi^g B(x) \sum_{j=1}^{G} \sigma_f^j(x)\tilde{\phi}^j(x) + \tilde{f}^g(x) \qquad g = 1, 2, \ldots, G \quad . \quad (9)$$

The $2N \times 2N$ square matrices $A^g(x)$ and $B(x)$ and the $2N$-vectors $\tilde{f}^g(x)$ result from the procedure described above, the $2N$-vectors $\tilde{\phi}^g(x)$ are composed of the unknown functions $\phi_i^{g\pm}(x)$.

$$\tilde{\phi}^g(x) = \begin{bmatrix} \phi_1^{g+}(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N^{g+}(x) \\ \phi_1^{g-}(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N^{g-}(x) \end{bmatrix}$$

$$A^g(x) = \sigma_t^g(x)\, C^{-1} H - \sigma_{sc}^{g\to g}(x) C^{-1} D$$

$$B(x) = C^{-1} D$$

$$C = \begin{bmatrix} C^{(1)} & C^{(2)} \\ C^{(3)} & C^{(4)} \end{bmatrix}, \quad c_{ij}^{(1)} = \int_{-1}^{1} \mu w_i^{+}(\mu)\psi_j^{+}(\mu)\, d\mu \quad , \quad c_{ij}^{(2)} = \int_{-1}^{1} \mu w_i^{+}(\mu)\psi_j^{-}(\mu)\, d\mu$$

$$c_{ij}^{(3)} = \int_{-1}^{1} \mu w_i^{-}(\mu)\psi_j^{+}(\mu)\, d\mu \quad , \quad c_{ij}^{(4)} = \int_{-1}^{1} \mu w_i^{-}(\mu)\psi_j^{-}(\mu)\, d\mu$$

$$H = \begin{bmatrix} H^{(1)} & H^{(2)} \\ H^{(3)} & H^{(4)} \end{bmatrix}, \quad h_{ij}^{(1)} = \int_{-1}^{1} w_i^{+}(\mu)\psi_j^{+}(\mu)\, d\mu \quad , \quad h_{ij}^{(2)} = \int_{-1}^{1} w_i^{+}(\mu)\psi_j^{-}(\mu)\, d\mu$$

$$h_{ij}^{(3)} = \int_{-1}^{1} w_i^{-}(\mu)\psi_j^{+}(\mu)\, d\mu \quad , \quad h_{ij}^{(4)} = \int_{-1}^{1} w_i^{-}(\mu)\psi_j^{-}(\mu)\, d\mu$$

$$D = \begin{bmatrix} D^{(1)} & D^{(2)} \\ D^{(3)} & D^{(4)} \end{bmatrix}, \quad d_{ij}^{(1)} = \frac{1}{2} \int_{-1}^{1} w_i^{+}(\mu)\, d\mu \int_{-1}^{1} \psi_j^{+}(\mu')\, d\mu' \quad , \quad d_{ij}^{(2)} = \frac{1}{2} \int_{-1}^{1} w_i^{+}(\mu)\, d\mu \int_{-1}^{1} \psi_j^{-}(\mu')\, d\mu$$

$$d_{ij}^{(3)} = \frac{1}{2} \int_{-1}^{1} w_i^{-}(\mu)\, d\mu \int_{-1}^{1} \psi_j^{+}(\mu')\, d\mu' \quad , \quad d_{ij}^{(4)} = \frac{1}{2} \int_{-1}^{1} w_i^{-}(\mu)\, d\mu \int_{-1}^{1} \psi_j^{-}(\mu')\, d\mu$$

$$\tilde{f}^g(x) = C^{-1} \begin{bmatrix} k_1^{g+}(x) \\ \cdot \\ \cdot \\ \cdot \\ k_N^{g+}(x) \\ k_1^{g-}(x) \\ \cdot \\ \cdot \\ \cdot \\ k_N^{g-}(x) \end{bmatrix}$$

$$k_i^{g\pm} = \frac{1}{2} \sum_{j=1}^{G} \sigma_s^{j\to g} \int_{-1}^{1} f^{j0}(x,\mu')\, d\mu' \cdot \int_{-1}^{1} w_i^{\pm}(\mu)\, d\mu \quad .$$

Thus, Eq. (9) is a system of linear first-order differential equations consisting of $2N \times G$ coupled differential equations. This system of equations can be considered as a reduced multigroup problem. So that Eq. (8a) satisfies homogeneous boundary conditions, we require the "new" boundary conditions, Eqs. (10), for the functions $\phi_i^{g\pm}(x)$.

$$\vec{\phi}^{\,g+}(d) = 0 \tag{10a}$$

$$\vec{\phi}^{\,g-}(0) = 0 \quad . \tag{10b}$$

The N-vectors $\vec{\phi}^{\,g\pm}(x)$ are given by Eq. (11).

$$\vec{\phi}^{\,g\pm}(x) = \begin{bmatrix} \phi_1^{g\pm}(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N^{g\pm}(x) \end{bmatrix} \quad . \tag{11}$$

The solution of Eqs. (9) and (10) can be obtained by approximating Eq. (9) by finite difference equations.[14] After reordering the resulting system of linear equations, each group equation can be solved by an iteration scheme similar to that described in Ref. 15 (inner iteration), and the whole system may be solved by the familiar source-iteration technique[14] (outer iteration). The coefficient matrix of the system of linear equations to be solved after reordering is shown schematically in Fig. 1, where the squares are $N \times N$ sub-matrices and $M = 2 \times (NX - 1)$. NX is the number of mesh points in the finite difference approximation.

C. Expansion in Functions of the Angle Variable and Functions of the Energy Variable

We expand the unknown solution as in Eq. (12)

$$f^{g1}(x,\mu) = \sum_{i=1}^{N} [\phi_i^+(x)\psi_i^+(\mu)\rho_i^{g+} + \phi_i^-(x)\psi_i^-(\mu)\rho_i^{g-}]$$
$$g = 1, 2, \ldots, G \quad . \tag{12}$$

The angle modes $\psi_i^{\pm}(\mu)$ and the energy modes $\vec{\rho}_i^{\,\pm}$, the latter in the form of multigroup spectra, are chosen by the user so that both sets are well suited to the problem to be solved. The $\psi_i^{\pm}(\mu)$ and the $\vec{\rho}_i^{\,\pm}$ must be linearly independent functions and G-vectors, respectively, and the $\psi_i^{\pm}$ must satisfy Eq. (8a) and (8b).

Substituting Eq. (12) into Eq. (7) we determine the coefficient functions $\phi_i^{\pm}(x)$ in the usual manner by the method of weighted residuals. In the multigroup model, multiplication with energy-dependent weighting functions $v_i^{\pm}(E)$ and integrating over E is equivalent to scalar multiplying with the weighting G-vectors $\vec{v}_i^{\,\pm}$. After some algebra we obtain Eq. (13), from which the coefficient functions $\phi_i^{\pm}(x)$ can be determined.

$$\frac{d}{dx}\vec{\phi}(x) + A(x)\vec{\phi}(x) = \vec{f}(x) \quad . \tag{13}$$

The $2N \times 2N$ square matrix $A(x)$ and the $2N$-vector $\vec{f}(x)$ are given as follows:

$$A = B^{-1}R$$

$$B = \begin{bmatrix} B^{(1)} & B^{(2)} \\ B^{(3)} & B^{(4)} \end{bmatrix} \quad \text{with} \quad b_{ij}^{(1)} = c_{ij}^{(1)} \cdot \sum_{g=1}^{G} v_i^{g+}\rho_j^{g+} \quad , \quad b_{ij}^{(2)} = c_{ij}^{(2)} \cdot \sum_{g=1}^{G} v_i^{g+}\rho_j^{g-}$$
$$b_{ij}^{(3)} = c_{ij}^{(3)} \cdot \sum_{g=1}^{G} v_i^{g-}\rho_j^{g+} \quad , \quad b_{ij}^{(4)} = c_{ij}^{(4)} \cdot \sum_{g=1}^{G} v_i^{g-}\rho_j^{g-}$$

$$R = \begin{bmatrix} R^{(1)} & R^{(2)} \\ R^{(3)} & R^{(4)} \end{bmatrix} \quad \text{with, e.g.,} \quad r_{ij}^{(3)} = h_{ij}^{(3)} \cdot \sum_{g=1}^{G} \sigma_t^g(x) v_i^{g-}\rho_j^{g+} - d_{ij}^{(3)} \cdot \sum_{g=1}^{G} v_i^{g-} \sum_{l=1}^{G} \sigma_s^{l\to g}(x)\rho_j^{l+}$$

$[c_{ij}^{(n)}, d_{ij}^{(n)}$, and $h_{ij}^{(n)}$ from part B$]$

$$\vec{f}(x) = B^{-1} \cdot \begin{bmatrix} k_1^+(x) \\ \cdot \\ \cdot \\ \cdot \\ k_N^+(x) \\ k_1^-(x) \\ \cdot \\ \cdot \\ \cdot \\ k_N^-(x) \end{bmatrix}$$

with

$$k_i^{\pm}(x) = \frac{1}{2}\sum_{g=1}^{G} v_i^{g\pm} \sum_{l=1}^{G} \sigma_s^{l\to g} \int_{-1}^{1} f^{l0}(x,\mu')\,d\mu' \cdot \int_{-1}^{1} w_i^{\pm}(\mu)\,d\mu \quad .$$

The $2N$-vector $\vec{\phi}(x)$ is composed of the unknown coefficient functions

$$\vec{\phi}(x) = \begin{bmatrix} \phi_1^+(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N^+(x) \\ \phi_1^-(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N^-(x) \end{bmatrix} \quad .$$

Equation (13) is a system of linear first-order differential equations consisting of only $2N$ coupled differential equations and can thus be considered as a reduced one-group problem. As before, we require some "new" boundary conditions, Eqs. (14), so that Eq. (12) satisfies homogeneous boundary conditions.

$$\vec{\phi}^{\,+}(d) = 0 \tag{14a}$$

$$\vec{\phi}^{\,-}(0) = 0 \quad . \tag{14b}$$

The N-vectors $\vec{\phi}^{\,\pm}$ are given analogous to Eq. (11). The solution of Eq. (13) may be obtained after approximating the differential equations by finite difference equations by the inner iteration scheme mentioned above,[14] since Eq. (13) has the same form as each of the G equations of Eq. (9).

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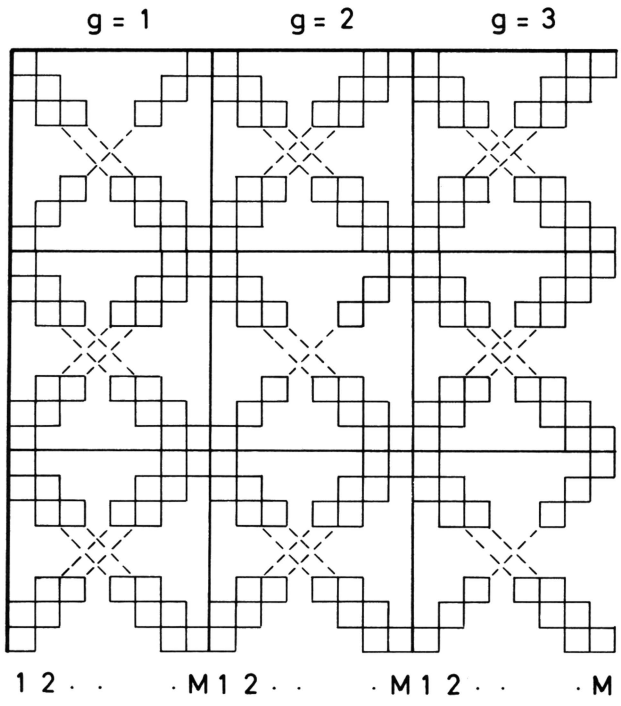

Fig. 1. Coefficient matrix of finite difference representation of Eq. (9).

D. Expansion in Functions of the Space Variable

As a third possibility we expand the solution of Eq. (7) in functions of the space variable,[1] Eq. (15a)

$$f^{g1}(x) = \sum_{i=1}^{N} [\phi_i^+(x)\psi_i^{g+}(\mu) + \phi_i^-(x)\psi_i^{g-}(\mu)] \qquad g = 1, 2, \ldots, G \quad . \quad (15a)$$

The space-dependent modes $\phi_i^{\pm}(x)$ are chosen as a set of linearly independent functions by the user, taking into account his knowledge of the desired solution. These modes are required to satisfy Eqs. (14), whereas the angle-dependent coefficient functions $\psi_i^{g\pm}$ are required to satisfy the conditions given in Eqs. (15b) and (15c)

$$\psi_i^{g+}(\mu) = 0 \qquad 0 < \mu \le 1 \qquad i = 1, 2, \ldots, N; \quad g = 1, 2, \ldots, G \quad (15b)$$

$$\psi_i^{g-}(\mu) = 0 \qquad -1 \le \mu \le 0 \qquad (15c)$$

According to the method of weighted residuals, we substitute our ansatz, Eqs. (15), into Eq. (7) and require orthogonality of the residuum in the interval [-1, 0] to the N weighting functions $w_j^+(x)$ and orthogonality in the interval [0, 1] to the N weighting functions $w_j^-(x)$. The weighting functions are to be chosen as before by the user as sets of linearly independent functions. With the scalar product

$$(f,g) = \int_0^d f(x)g(x)dx$$

we obtain Eqs. (16) from which the half-range coefficient functions $\psi_i^{g\pm}(\mu)$ can be determined.

$$A^{g+}(\mu)\tilde{\psi}^{g+}(\mu) = \frac{1}{2}B^+ \sum_{j=1}^{G} \sigma_s^{j\to g} \int_{-1}^{0} \tilde{\psi}^{j+}(\mu')d\mu' + \frac{1}{2}D^- \times \sum_{j=1}^{G} \sigma_s^{j\to g} \int_0^1 \tilde{\psi}^{j-}(\mu')d\mu' + \tilde{f}^{g+}(\mu)$$
$$-1 \le \mu \le 0 \; ; \; g = 1, 2, \ldots, G \quad . \quad (16a)$$

$$A^{g-}(\mu)\tilde{\psi}^{g-}(\mu) = \frac{1}{2}D^+ \sum_{j=1}^{G} \sigma_s^{j\to g} \int_{-1}^{0} \tilde{\psi}^{j+}(\mu')d\mu' + \frac{1}{2}B^- \times \sum_{j=1}^{G} \sigma_s^{j\to g} \int_0^1 \tilde{\psi}^{j-}(\mu')d\mu' + \tilde{f}^{g-}(\mu)$$
$$0 < \mu \le 1 \; ; \; g = 1, 2, \ldots, G \quad . \quad (16b)$$

The $N \times N$ square matrices $A^{g\pm}(\mu)$, $B^{\pm}$ and $D^{\pm}$ and the N-vectors $\tilde{f}^{g\pm}(\mu)$ result from the procedure described above. The elements are given as follows:

$$a_{ij}^{g\pm} = \mu\left(w_i^{\pm}, \frac{d}{dx}\phi_j^{\pm}\right) + (w_i^{\pm}, \sigma_t^g\phi_j^{\pm})$$
$$b_{ij}^{\pm} = (w_i^{\pm}, \phi_j^{\pm})$$
$$d_{ij}^{\pm} = (w_i^{\mp}, \phi_j^{\pm})$$
$$f_i^{g\pm} = \frac{1}{2}\left(w_i^{\pm}, \sum_{l=1}^{G} \sigma_s^{l\to g} \int_{-1}^{1} f^{lo}(x,\mu')d\mu'\right) \quad .$$

The N-vectors $\tilde{\psi}^{g\pm}(\mu)$ are given analogous to Eq. (11).

If the inverses of $A^{g+}(\mu)$ and $A^{g-}(\mu)$ exist in the interval [-1, 0] and in [0, 1], respectively, we can multiply Eqs. (16a) and (16b) from the left with these inverses and obtain a system of Fredholm integral equations of the second type with degenerate kernel. Such systems can be reduced to systems of linear equations by integrating over μ. Each group equation of the resulting system can be solved directly (no inner iteration) and the whole system can be solved by the source-iteration scheme.

E. Expansion in Functions of the Space Variable and Functions of the Energy Variable

We choose the same expansion as is given by Eq. (12), but this time the space modes $\phi_i^{\pm}(x)$ and the energy modes $\tilde{\rho}_i^{\pm}$ are to be specified by the user, whereas the half-range coefficient functions $\psi_i^{\pm}(\mu)$ are to be determined. The coefficient functions again are restricted to half-range functions, according to Eqs. (8), and the expansion functions $\phi_i^{\pm}(x)$ are required to satisfy the boundary conditions, Eqs. (14). In this way, our ansatz, Eq. (12), satisfies homogeneous boundary conditions. Proceeding as before with the method of weighted residuals, we find the following system of equations, Eqs. (17), for determining the coefficient functions $\psi_i^{\pm}(\mu)$.

$$A^+(\mu)\tilde{\psi}^+(\mu) = B^+ \int_{-1}^{0} \tilde{\psi}^+(\mu')\,d\mu' + D^- \int_0^1 \tilde{\psi}^-(\mu')\,d\mu' + \tilde{f}^+(\mu) \qquad -1 \le \mu \le 0 \quad (17a)$$

$$A^-(\mu)\tilde{\psi}^-(\mu) = D^+ \int_{-1}^{0} \tilde{\psi}^+(\mu')\,d\mu' + B^- \int_0^1 \tilde{\psi}^-(\mu')\,d\mu' + \tilde{f}^-(\mu) \qquad 0 < \mu \le 1 \quad . \quad (17b)$$

The elements of the $N \times N$ square matrices $A^{\pm}(\mu)$, $B^{\pm}$, $D^{\pm}$ and of the N-vectors $\tilde{f}^{\pm}(\mu)$ are given by the following equations:

$$a_{ij}^{\pm}(\mu) = \mu\left(w_i^{\pm}, \frac{d}{dx}\phi_j^{\pm}\right) \cdot \sum_{g=1}^{G} v_i^{g\pm}\rho_j^{g\pm} + \left(w_i^{\pm}, \sum_{g=1}^{G} \sigma_t^g v_i^{g\pm}\rho_j^{g\pm}\phi_j^{\pm}\right)$$

$$b_{ij}^{\pm} = \frac{1}{2}\left(w_i^{\pm}, \sum_{g=1}^{G} v_i^{g\pm} \sum_{l=1}^{G} \sigma_s^{l\to g}\rho_j^{l\pm}\phi_j^{\pm}\right)$$

$$d_{ij}^{\pm} = \frac{1}{2}\left(w_i^{\mp}, \sum_{g=1}^{G} v_i^{g\mp} \sum_{l=1}^{G} \sigma_s^{l\to g}\rho_j^{l\pm}\phi_j^{\pm}\right)$$

$$f_i^{\pm} = \frac{1}{2}\left(w_i^{\pm}, \sum_{g=1}^{G} v_i^{g\pm} \sum_{l=1}^{G} \sigma_s^{l\to g} \int_{-1}^{1} f^{lo}(x,\mu')d\mu'\right) \quad .$$

The $2N$ weighting functions $w_i^{\pm}(x)$ and the $2N$ weighting G-vectors $\check{v}_i^{\pm}$ are chosen as linearly independent sets by the user and, as pointed out before, determine in which sense the approximate solution Eq. (12) will be a "best" approximation. The two N-vectors $\check{\psi}^{\pm}(\mu)$ are constituted by the unknown half-range coefficient functions $\psi_i^{\pm}(\mu)$. If, again, the inverses of $A^+(\mu)$ and $A^-(\mu)$ exist in $[-1, 0]$ and $[0, 1]$, respectively, we can reduce Eqs. (17) as described before to a system of linear equations, which consists of only $2N$ linear equations and therefore, in general, can be solved directly (no iteration).

F. Alternative Treatment of Boundary Conditions

Instead of dividing the neutron flux into an uncollided and a scattered part as in Eq. (4) and then approximately solving for the scattered part with homogeneous boundary conditions we can proceed in a different manner. We expand the total unknown solution of Eq. (1) as in Eqs. (8a), (12), or (15a). So that these expansions satisfy Eqs. (2) and (3), we require additional or modified conditions for the modes and coefficient functions. The simplest case is the expansion Eq. (15a). If, in addition, we require Eq. (18a) for the space-dependent modes and if we modify the restrictions Eqs. (15b) and (15c) for the coefficient functions to those given by Eqs. (18b)-(18d), Eq. (15a) satisfies Eqs. (2) and (3)

$$\phi_1^+(0) = 1 \tag{18a}$$

$$\psi_i^{g+}(\mu) = 0 \qquad 0 < \mu \leq 1\,; \qquad i = 2, 3, \ldots, N \tag{18b}$$

$$\psi_i^{g-}(\mu) = 0 \qquad -1 \leq \mu \leq 0\,; \qquad i = 1, 2, \ldots, N \tag{18c}$$

$$\psi_1^{g+}(\mu) = r_l^g(\mu) \qquad 0 < \mu \leq 1 \tag{18d}$$

$$g = 1, 2, \ldots, G \quad .$$

The resultant reduced equations, Eqs. (16), will become the same with the exception of the N-vectors $\check{f}^{g\pm}(\mu)$, which will change.

In the three other expansions a similar way for satisfying the boundary conditions Eqs. (2) and (3) in general is not possible. However, if the angular distribution of the incident neutrons is the same in all energy groups,

$$r_l^g(\mu) = p^g r_l(\mu)$$

where the G-vector $\check{p}$ describes the energy distribution of the incident neutrons, we simply can require the following additional and modified conditions.

Additional condition for the expansions Eqs. (8a) and (12) (angle-dependent modes)

$$\psi_1^-(\mu) = r_l(\mu) \qquad 0 < \mu \leq 1 \quad .$$

Additional conditions for the expansion Eq. (12) (angle-dependent modes)

$$\rho_1^{g-} = p^g \qquad g = 1, 2, \ldots, G \quad .$$

Modified conditions for the expansion Eq. (8a)

$$\phi_1^{g-}(0) = p^g \qquad g = 1, 2, \ldots, G \quad .$$

Modified condition for the expansion Eq. (12) (angle-dependent modes)

$$\phi_1^-(0) = 1 \quad .$$

Additional conditions for the expansion Eq. (12) (space-dependent modes)

$$\phi_1^+(0) = 1$$
$$\rho_1^{g+} = p^g \qquad g = 1, 2, \ldots, G \quad .$$

Modified conditions for the expansion Eq. (12) (space-dependent modes)

$$\psi_1^+(\mu) = r_l(\mu) \qquad 0 < \mu \leq 1 \quad .$$

The resultant reduced equations retain the form given above, only the inhomogeneous terms $\check{f}$ will change. Therefore, the methods of solution mentioned above can be applied as before.

If the boundary conditions are treated in this way, care has to be taken in choosing the space-dependent mode $\phi_1^+(x)$ because now the uncollided part of the neutron flux is described by the products $\phi_1^+(x)\psi_1^{g+}(\mu)$ or $\phi_1^+(x)\psi_1^+(\mu)\tilde{\rho}_1^+$ which do not have any degree of freedom and therefore will contain errors which strongly depend on the mode $\phi_1^+(x)$. The total neutron flux, however, does not depend so strongly on the mode $\phi_1^+(x)$ because the other terms of the expansion to a certain degree will compensate for the errors introduced by $\phi_1^+(x)$.

G. Discussion of Restrictions

Some of the restrictions mentioned above are not necessary. First, the assumption of isotropic scattering in the LS can be dropped. The consequence is more computational effort for the evaluation of the scattering integral. In the case of expansion in functions of the space variable the simple structure of the resultant integral equations is lost. The flexibility of the expansions is enlarged if the space- or angle-dependent modes and perhaps even the weighting functions can be chosen different from energy group to energy group. This will reduce the total number of modes required in each group but, on the other hand, will result in more group-dependent matrices and thus will require more storage during computation. Besides, it then becomes possible to treat any boundary condition in the manner described in part F of this section.

The flexibility of the expansions is also enlarged if the modes and weighting functions can be chosen to be different in different zones of a multilayer problem or in the different regions of a homogeneous thick slab. This will introduce discontinuities at the interfaces where the modes and weighting functions change and will require some further conditions for controlling such discontinuities.[3] While the errors near the interfaces will probably become greater, a reduction of the overall error or of the necessary number of modes can be expected.

III. NUMERICAL EXAMPLE

We have applied the methods described above to calculating the neutron distribution within a thin (10 cm) homogeneous slab consisting of natural uranium with incident neutrons on one side. We assume that the energy distribution of the incident neutrons is equal to that of fission neutrons and that the angular distribution is isotropic in all groups. The total number of energy groups was 26.[13]

A. Angle Modes

While in the upper energy groups the angular distribution of the neutrons contains a strong forward component because of the incident neutrons, in the lower groups with no incident neutrons the angular distribution is rather isotropic in the middle of the slab and forward and backward peaked at the boundaries. Therefore, as the angle modes we have chosen one constant, one function containing a forward component and one function containing a backward component. The calculated angular distribution

of the neutrons at the two outer boundaries of the slab is shown in Figs. 2 and 3. For comparison, the results of an $S16$ calculation[16] are presented also. With only three angle modes in both half spaces which take into account only the "physical feeling," we obtain rather accurate results. The computing time required is much less than that required for the $S16$ calculation.

B. Angle and Energy Modes

Taking the same angle modes as before and choosing as the energy modes three spectra, the fission spectrum and two spectra obtained with a 26-group diffusion calculation at the center and near the upper surface of a natural uranium cylinder (R = 20 cm, H = 40 cm), the lower part of which was assumed to have independent volume sources for fission neutrons, we obtain the results shown in Figs. 4 and 5. Again, the results of the $S16$ calculation are presented for comparison. With very little computational effort (not more than for a multigroup diffusion calculation) we again obtain rather accurate results. The main error here seems to be introduced by the energy modes.

C. Space Modes

Expecting a more or less exponential behavior of the neutron flux, we have chosen as the space modes in both half spaces four functions given by the formula

$$\phi^{\pm}(x) = \exp(\pm\alpha x) - b^{\pm}(x) \quad .$$

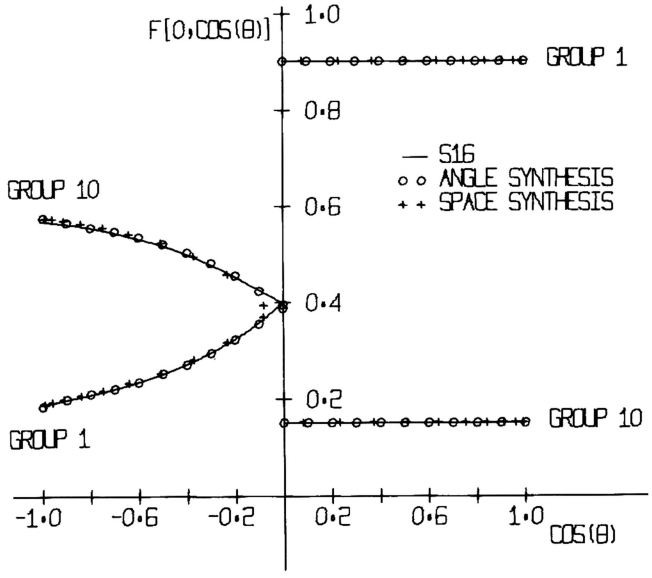

Fig. 2. Angular distribution on the left-hand side. Angle synthesis. Space synthesis.

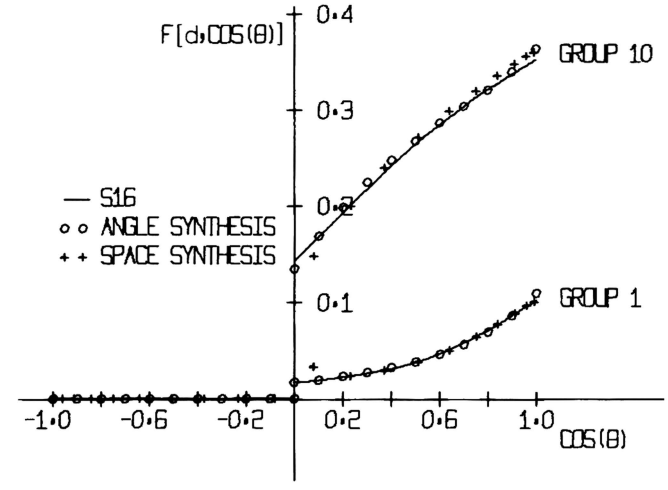

Fig. 3. Angular distribution on the right-hand side. Angle synthesis. Space synthesis.

[16]K. D. LATHROP, "DTF IV, a FORTRAN-IV Program for Solving the Multigroup Transport Equation with Anisotropic Scattering," LA-3373, Los Alamos Scientific Laboratory (1965).

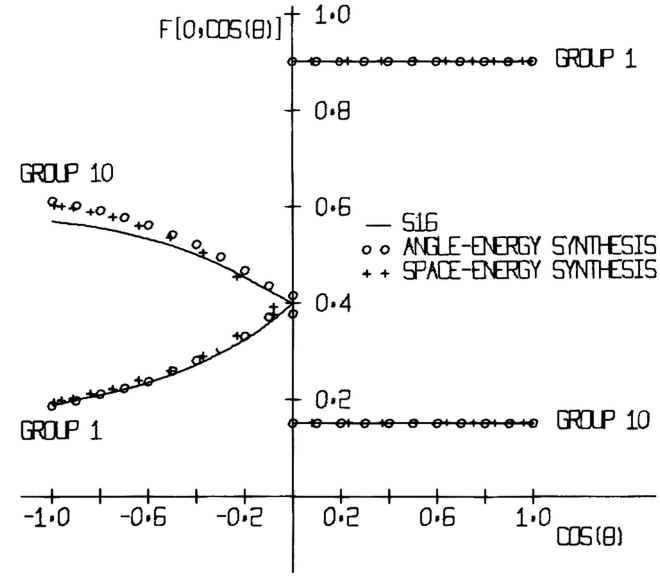

Fig. 4. Angular distribution on the left-hand side. Angle-energy synthesis. Space-energy synthesis.

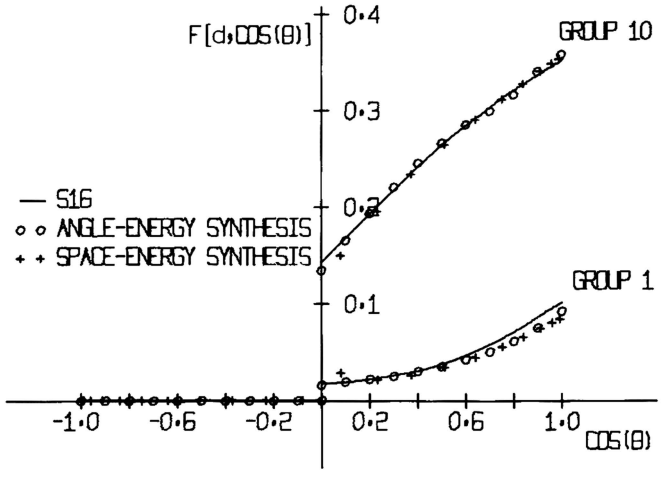

Fig. 5. Angular distribution on the right-hand side. Angle-energy synthesis. Space-energy synthesis.

To describe the flux distribution in all energy groups we have taken rather arbitrarily the following exponents

$$\alpha_1 \approx 1.5 \quad \sigma_{t,max}$$
$$\alpha_2 \approx \quad \sigma_{t,max}$$
$$\alpha_3 \approx \quad \sigma_{t,min}$$
$$\alpha_4 \approx -0.3 \quad \sigma_{t,min} \quad .$$

Calculations with other sets of constants in this range gave nearly identical results. The slowly varying functions $b(x)$ were chosen to satisfy the boundary conditions. Figures 2 and 3 show the results which agree rather well with those of the *S*16 calculation. The computing time required was about that of a multigroup diffusion calculation.

D. Space and Energy Modes

Taking the space and energy modes from part B and C of this section we obtain the results shown in Figs. 4 and 5. The accuracy again is rather good; only for small μ (i.e., nearly parallel to the slab surface) do the errors become considerable. This perhaps can be understood when remembering that the exponential behavior in these directions can only be described by rather large exponents. For μ close to unity the errors seem to be due to the energy modes as was stated before. The computing time required was less than that of the multigroup diffusion calculation.

IV. SUMMARY

This paper describes some special half-range expansions in known expansion functions and unknown coefficient functions for solving the multigroup Boltzmann equation for slabs. The expansion functions, to be chosen by the user, can take into account nearly all information that may be available over the space-, angle-, and energy-dependence of the expected solution. Transport boundary conditions can be exactly satisfied, which is of importance in small, absorbing or only weakly multiplying systems. Two of the expansions eliminate one, the other two expansions eliminate two variables. The resultant reduced equations for the coefficient functions are rather simple and can be solved numerically.

The expansions have been applied to calculations of the neutron distribution within a thin homogeneous natural uranium slab. It was demonstrated that results for such relatively simple problems can be obtained at rather little computational effort and are nearly as accurate as those of *SN* calculations of high order. It was further shown that even approximations of space-angle-energy-synthesis type, which eliminate two of three variables and result in very simple equations for the coefficient functions, are rather accurate. For this type of approximation the computing time required is about that of a multigroup diffusion calculation.

The flexibility of these expansions can be enlarged by using different sets of modes and weighting functions for each zone in multilayer problems or by using different sets of space or angle modes and weighting functions in different energy groups. In the latter case, however, the total computational effort may become greater even if the number of necessary modes can be reduced. Anisotropic scattering can be taken into account, but this also will increase the computational effort.

ACKNOWLEDGMENTS

The author wishes to thank H. Küsters for his encouragement of this work, for valuable discussions, and critical comments. Thanks are also due to W. Kofink and K. Wirtz for critical comments and for their interest in this work.

This manuscript is based on work accepted as a thesis by the University of Karlsruhe, Germany, Department of Engineering.