TABLE I

Iterative Solutions vs Exact Solution*

$$E_c = \gamma \int_0^\infty \varphi(\tfrac{1}{2}, t')\, dt'$$

$$\text{Percent Difference} = \left| \frac{\text{Exact} - \text{Approximate}}{\text{Exact}} \right| \times 100$$

	E_c	E_c[a]		E_c[b]		E_c[c]	
$(B/\pi)^2$	Exact (Ref. 2)	Value	Percent Diff.	Value	Percent Diff.	Value	Percent Diff.
1.2	0.235	0.2349	0.085	0.2349		0.2349	
2	1.162	1.166	0.172	1.1616		1.1622	
3	2.297	2.306	0.391	2.2960		2.3002	0.051
4	3.408	3.395	0.381	3.4141	0.179	3.4276	0.575
7	6.640	6.637	0.045	6.8136	2.614	6.9311	4.384
8	7.693	7.720	0.376	8.0190	4.238	8.2224	6.881
10	9.774	9.921	1.504	10.6446	8.906	11.2024	14.615

*In Eq. (1) α is taken as $2\pi^2$ as in Ref. 2.
[a]Taken from Ref. 1 supposedly calculated from Eqs. (3) and (4).
[b]Values obtained by us using Eqs. (3) and (4). The values of E_c obtained here are correct to four decimal places.
[c]Values obtained from Eqs. (9) and (10).

$$L_S\theta_3 = L_T\theta_2 + \alpha\gamma\left[\theta_2\int_0^t \theta_0 dt' + \theta_1\int_0^t \theta_1 dt' + \theta_0\int_0^t \theta_2 dt'\right] ,$$

etc.,

where the nonlinear terms are written explicitly. In Shotkin's iterative scheme the nonlinear term

$$\alpha\gamma\theta_1\int_0^t \theta_1 dt' \tag{8}$$

was included along with the terms

$$\alpha\gamma\theta_0\int_0^t \theta_1 dt' \text{ and } \alpha\gamma\theta_1\int_0^t \theta_0 dt'$$

in his second iteration. In our scheme the term (8) is of a different order as shown above. Since the last term of both Eqs. (3) and (4) arose from (8) this may explain why these last terms are of different order compared to the others in the second bracket of Eqs. (3) and (4).

In our scheme, if the term (8) is excluded we obtain, to order ϵ^2,

$$\left(\frac{B}{\pi}\right)^2 - 1 = C_1 E - \frac{C_3 D_1}{4} E^2 , \tag{9}$$

$$E_c = E + \frac{C_3}{8}E^2 + \frac{C_3}{64}\left[\left(\frac{B}{\pi}\right)^2 - 1\right]E^2 - \frac{C_3 D_3}{32}E^3 , \tag{10}$$

which are identical to Eqs. (3) and (4), respectively, except for the absence of the last terms. In this scheme the E-equation [that is Eq. (9)] is found by adding successively the coefficients of the $\sin\pi x$ mode from the right-hand side of all iterations up to the highest order iteration attempted and equating the sum to zero. This has its origin in the same argument used by Shotkin that the "secular" terms in the spatial mode expansion, that is terms in $\sin\pi x$ in the right-hand side of Eq. (5) [or Eq. (6) since $\epsilon = 1$], must be zero. Since Eq. (6) is broken-up into the different iteration equations, to obtain all secular terms up to the highest order iteration attempted, we must sum all these from the different iterations.

Table I shows the results calculated from Eqs. (9) and (10). We note that the percentage difference with the exact values is not drastically different from those calculated from Eqs. (3) and (4). Equations (9) and (10) still consider only the spatial modes up to $\sin 3\pi x$ in the iterations. In our scheme if a third order iteration is attempted it may be necessary to go up to $\sin 5\pi x$ mode and higher in all the iterations.

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July 24, 1969

Reply to Comments on the Iterative Approach to a Space-Time Nonlinear Problem

In response to the above Letter of Ibarra,[1] there was a numerical error on my part in Ref. 2. On examining my notes, I found that in calculating the quantity $C_3^2\, G_1/64$ in Eq. (46) of Ref. 2, I had correctly written $(0.16976)^2/64\,[72/(35\pi)]$ for the individual elements in this term but had somehow obtained 6.552×10^{-4} instead of the correct value 2.948×10^{-4}. On recalculation with this corrected value, I obtain agreement with the results of Ibarra.[1] These are shown in Table I, columns 2 and 3. These new values do not change any of the conclusions of Ref. 2. The one change that should be made (in addition to the corrections for the first iteration) is that in the paragraph after Eq. (47), the statement "The results . . . are seen to be in good agreement with the exact answers," should now read, "The results . . . are seen to be within 10% of the exact answers."

Although these percentage differences are an improvement on those obtained using a modal expansion,[3] they are

[1]H. IBARRA, "Comments on the Iterative Approach to a Space-Time Nonlinear Problem," *Nucl. Sci. Eng.*, **39**, 130 (1969).
[2]L. M. SHOTKIN, *Nucl. Sci. Eng.*, **36**, 97 (1969).
[3]J. CANOSA, *Nucl. Sci. Eng.*, **32**, 156 (1968).

TABLE I

Space-Time Problem: Comparison of Analysis with Computer Results for the Energy at the Core Center,

$$E_c = \gamma \int_0^\infty \phi(\tfrac{1}{2}, t')\, dt'$$

$$\text{Percent Difference} = \frac{\text{Exact - Approximate}}{\text{Exact}} \times 100$$

Column Number	1	2	3	4	5	6	7
				Percent Difference in Second-Order Iteration			
	Exact Computation (Ref. 3)	First-Order Iteration: $0(E^4)$		$0(E^4)$		$0(E^8)$	
$(B/\pi)^2$		Eq. (47) of Ref. 2	% Dif.	sin $3\pi x$, Eq. (4)	sin $5\pi x$, Eq. (7)	sin $3\pi x$, Eq. (8)	sin $5\pi x$, Eq. (9)
1.2	0.235	0.235	0.00	0.00	0.00	0.00	0.00
2	1.162	1.162	0.00	+0.09	0.00	+0.09	0.00
3	2.297	2.296	+0.04	+0.44	+0.17	+0.31	+0.04
4	3.408	3.414	-0.18	+0.85	+0.41	+0.44	+0.03
7	6.640	6.814	-2.62	+2.59	+1.67	+0.84	-0.15
8	7.693	8.019	-4.24	+3.01	+1.95	+0.97	-0.23
10	9.774	10.644	-8.91	+3.12	+1.80	+1.32	-0.31

not as good as originally advertised.[2] In trying to improve the answers, Ibarra has used the perturbation method we used [compare Eqs. (32) of Ref. 2 and Eq. (6) of Ref. 1] but in introducing an additional restriction [Eq. (7) of Ref. 1] he omits a small, but important, nonlinear term [Eq. (8) of Ref. 1]. As Ibarra correctly shows in the last column of his Table I, this omission gives poorer accuracy. The power of the perturbation method used in Ref. 2 is that higher-order terms in the time-dependent expansion can be obtained exactly and with minimum effort at each order of iteration. Thus, leaving some of them out should lead to poorer accuracy. It is in the space-dependent expansion that the major truncation-simplification is introduced. In the problem studied the space series was truncated at the sin $3\pi x$ mode. At a given order of iteration, including higher-order spatial modes in an exact fashion becomes a somewhat more cumbersome procedure than including higher-order terms in the time-dependent series.

It is a simple matter to improve the accuracy by going ahead one more iteration; a route that was not followed in Ref. 2 only because reasonable accuracies were incorrectly calculated at the first iteration. In going to the second iteration, we must solve for ϕ_3 in the equation (see Ref. 2 for notation)

$$L_s\phi_3 = L_T\,\phi_2 + N(\phi_2) \quad , \tag{1}$$

where

$$\phi_2 = A \sin \pi x + B_3 \sin 3\pi x \tag{2}$$

with

$$B_3(t) = -\frac{1}{8\pi^2} \times \left\{[-(B^2 - \pi^2)\, M_3 + \alpha\gamma C_3]\, A \int A dt + \alpha\gamma M_3 D_3 F_1(t) + \alpha\gamma M_3^2\, C_3 F_2(t)\right\} \quad .$$

Here

$$M_3 = \frac{\alpha\gamma\, C_3}{(1 - 3^2)\pi^2} \quad ;$$

note that the factor of π^2 was omitted in the misprinted M_n after Eq. (38) in Ref. 2.

Since $F_2(t)$ is of $0(E_4)$, $F_2 \int F_2 dt$ in $N(\phi_2)$ will lead to terms of $0(E^8)$ in the second iteration—Eq. (1). As a first approximation, let us compute the changes caused by the second iteration only in those coefficients of E^n terms that occur in the first iteration. That is, only terms up to and including E^4 in the expression for E_c. In this way, Eqs. (46) and (47) of Ref. 2 are changed to

$$\left(\frac{B}{\pi}\right)^2 - 1 = C_1 E - \frac{C_3 D_1}{4}(1 + \Delta)E^2 + \left[\frac{C_3 G_1}{4}(1 + \Delta)^2 + D_1 D_3\right] \frac{C_3 E^3}{16} \tag{3}$$

$$E_c = E + \frac{C_3}{8}(1 + \Delta + \Delta^2)E^2 - \frac{C_3 D_3}{32}(1 + 2\Delta)E^3 + \left[\frac{C_3 D_3^2}{144} + \frac{G_3 C_3^2}{512}(1 + 3\Delta + \Delta^2)\right] E^4 \tag{4}$$

with

$$\Delta \equiv \frac{\left(\frac{B}{\pi}\right)^2 - 1}{8} \quad . \tag{5}$$

The percent difference from the exact solution computed with these equations is shown in column 4 of Table I. The accuracy when $(B/\pi)^2 = 10$ is improved from 8.91 to 3.12%. Further improvement may be obtained at this order of approximation by including the effect of the sin $5\pi x$ mode. This can be done by adding the term

$$M_5 \left(1 + \frac{\Delta}{3}\right) \sin 5\,\pi x\, A \int A dt \tag{6}$$

to the expression for ϕ_2 in Eq. (2). This term comes from including the sin $5\pi x$ mode in the ϕ_1 solution[2] and computing terms of $0(A^2)$ in ϕ_2 (hence, the $\Delta/3$ factor). One then gets an additional term, $(-C_5/24)(1 + \Delta/3)E^2$, to add to Eq. (4):

$$E_c\Big|_{\sin 5\pi x} = E_c\Big|_{\text{Eq. (4)}} - \frac{C_5}{24}\left(1 + \frac{\Delta}{3}\right)E^2 \quad . \tag{7}$$

The improved accuracies obtained using Eq. (7) are shown in column 5 of Table I. As these solutions now give accuracies comparable to those advertised in Ref. 2, we

could stop here. However, it is a simple matter of multiplication and addition to include up to $0(E^8)$ terms in this second iteration and improve the accuracy even further.

We thus obtain the expression for E_c to $0(E^8)$:

$$E_c = E_c\Big|_{[\text{from Eq. (4)}]} - \frac{C_3^2 G_3 D_3}{1024}\left(\frac{3}{2}+\Delta\right)E^5 + \frac{G_3 C_3^2}{8192}\left[-\frac{C_3 G_3}{2}(1+\Delta)+D_3^2\right]E^6 + \frac{G_3^2 D_3 C_3^3}{8(8192)}E^7 + \frac{G_3^3 C_3^4}{8(262,144)}E^8 \quad . \tag{8}$$

A similar expression is obtained for the right-hand side of Eq. (3) up to $0(E^7)$. The improved accuracies computed by including these higher-order terms in the second iteration are shown in column 6 of Table I. In column 7 we show the accuracy if one includes the $\sin 5\pi x$ correction

$$E_c = E_c\Big|_{[\text{from Eq. (8)}]} - \frac{C_5[1+(\Delta/3)]}{24}E^2 \quad . \tag{9}$$

Thus, reasonable accuracy can be obtained in the second iteration for this problem.

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Comments on Theoretical and Experimental Criteria for Reactor Stability

Kalinowski[1] recently criticized the proof of a stability criterion given by Gyftopoulos.[2] But neither the arguments given by Kalinowski nor the reply by Gyftopoulos are satisfying because the solutions of the corresponding kinetic equations [Eqs. (1) through (3) in Ref. 2] are interpreted in a finite dimensional Euclidean state space. Since these equations represent a system of functional-differential equations, it is necessary to interprete the solutions in an appropriate function space [in this case $C(-\infty, 0]$].

Indeed Eqs. (1) through (3) of Ref. 2 are autonomous as Gyftopoulos says. This is due to the fact that (cf., Ref. 3, p. 764)

$$I = \int_{-\infty}^{t} f(t-\tau)p(\tau)d\tau = \int_{-\infty}^{0} f(-\tau)p(t+\tau)d\tau \quad .$$

$p(t+\tau)$, $-\infty < \tau \leq 0$, is a function in $(-\infty, 0]$ usually denoted by the symbol p_t. For any value of t the function p_t belongs to the space $C(-\infty, 0]$. Therefore, the integral I can be written as $I = F(p_t)$. If t varies, then I changes its value only if p_t varies as an element of $C(-\infty, 0]$.

The Liapunov functional V used by Gyftopoulos [Eq. (17) in Ref. 2], contrary to the statement of Kalinowski, is positive definite without any assumption over the integrals, if the given conditions on the parameters are fulfilled. If the calculations relating to the step from Eq. (19) to Eq. (20) of Ref. 2 were correct, then the time derivative of V would be only negative semidefinite and not, as Gyftopoulos says, negative definite. In fact, $\dot{V}$ is zero for $p(t) = 0$ and $c_i(t)$ $(i = 1, \ldots, m)$ arbitrarily. But apart from the correctness of Eq. (20), which will be discussed below, the methodological foundation of the paper by Gyftopoulos is wrong, because it is not possible to apply classical theorems of Liapunov's direct method to Eqs. (1) through (3) of Ref. 2, since these are functional-differential equations and not ordinary differential equations. In Ref. 3, considering Eq. (20) as correct, it is shown that Gyftopoulos' criterion can be proved applying an extension of Liapunov's direct method to functional-differential equations given by Hale (Ref. 4). Moreover, since Gyftopoulos provides no proof for the domain of asymptotic stability given in Ref. 2 [in fact the domain defined by the inequalities (23) is merely a domain where V is positive definite], in Ref. 3 there is given a domain which is surely contained in the domain of attraction relating to the power equilibrium state.

Some correspondence following up the publication of the paper quoted in Ref. 3 revealed an error in the step from Eqs. (A7) to (A9) in Ref. 2. Precisely, one has that from

$$|K(\omega_1,\omega_2)|^2 = 4\text{Re}\,G(j\omega_1)\text{Re}\,G(j\omega_2) + C^2(\omega_1,\omega_2) \tag{1}$$

does not follow

$$K(\omega_1,\omega_2) = 2[\text{Re}\,G(j\omega_1)\text{Re}\,g(j\omega_2)]^{1/2} + jC(\omega_1,\omega_2) \quad , \tag{2}$$

but more generally

$$(\omega_1,\omega_2) = 2[\text{Re}\,G(j\omega_1)\text{Re}\,G(j\omega_2)]^{1/2}\alpha(\omega_1,\omega_2) + C(\omega_1,\omega_2)\beta(\omega_1,\omega_2) \quad . \tag{3}$$

$\alpha(\omega_1,\omega_2)$ and $\beta(\omega_1,\omega_2)$ are complex numbers with $|\alpha| = |\beta| = 1$. In other words, knowning $|K(\omega_1,\omega_2)|^2$ one can not determine uniquely $K(\omega_1,\omega_2)$.

The incorrectness of Eq. (20) in Ref. 2 is confirmed by the discussion of a particular case. For instance Gyftopoulos considers the case where b is very large. In fact, considering, as Gyftopoulos says, only the last integral term in V we have

$$V = \frac{b\lambda}{a}\int_{-\infty}^{t}\frac{a-1-p(\tau)}{1+p(\tau)}k^2(\tau)\,d\tau \tag{4}$$

and

$$\dot{V} = \frac{b\lambda}{a}\times\frac{a-1-p(t)}{1+p(t)}k^2(t) \quad . \tag{5}$$

Now, if we have $d^2 = a$ and $a, d > 1$, then

$$-1 < p(t) < d-1 \tag{6}$$

implies

$$-1 < p(t) < a-1 \quad . \tag{7}$$

Equations (5) and (7) prove that $\dot{V}$ is positive semidefinite, because b, a, λ are positive constants. But according to Eq. (20) of Ref. 2, $\dot{V}$ should be negative semidefinite!

It is interesting to mention that in the special case $b = 0$ the condition $\text{Re}\,G(\omega) > 0$ given by Gyftopoulos coincides with the condition relating to the new stability criterion given by the authors of this letter in Ref. 5, considering a new Liapunov functional and applying an extension of the stability theory given by Hale in Ref. 4. However the coincidence of $G(\omega) > 0$ with the new criterion in the

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[4] J. K. HALE, *J. Diff. Eqs.*, **1**, 452 (1965).
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