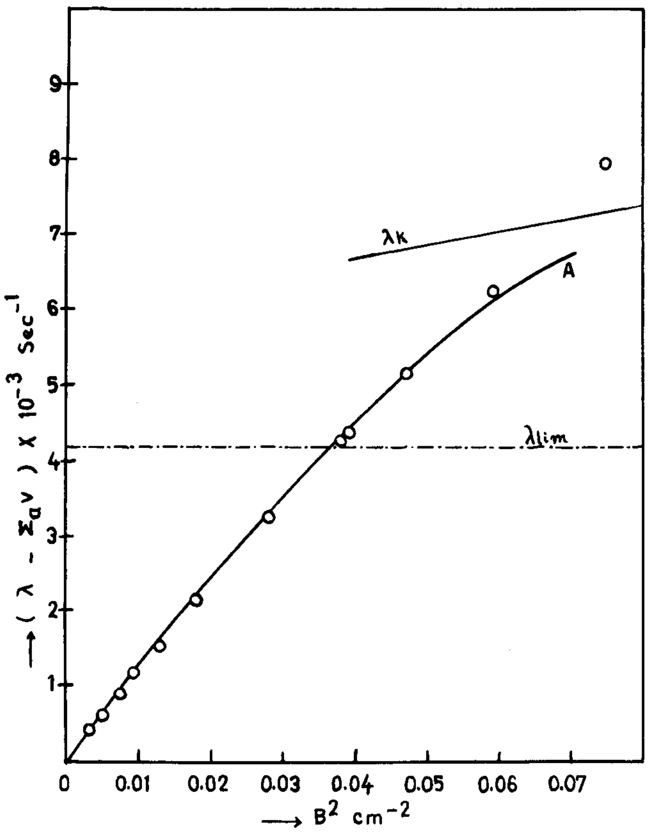

Fig. 1. Curve A shows the calculated values of $(\lambda - \Sigma_a v)$ as a function of B^2 for beryllium at room temperature. The broken line shows λ_{lim} whereas the full straight line gives λ_K. Circles are the experimental points of Andrews[8].

We would like to mention at the end that the eigenvalues of the complete Eq. (2) without the cutoff in energy will dominate only after long times but by then most of the neutrons would have leaked out.

Details of this work are to be reported shortly.

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Decay Constant of a Neutron Pulse

Dr. Kothari[1] has shown that if one uses a particular model for the scattering of neutrons by beryllium, and works with a cut-off scale of velocities, one will obtain the values of $\lambda_0(B^2)$ measured by Andrews. This is an interesting result. It is not an explanation, because it rests upon the ad hoc notion that one *should* limit the range of neutron velocities. We are aware of no physical principle or experimental constraint that compels one to cut-off at ten degrees, or at five degrees, especially when the experiment is marked by a strong diffusion-cooling effect. If the physicist observes an exponential decrease characterized by $\lambda > \lambda^* = (v\Sigma)_{\min}$, the phenomenon must be understood through reasoning based upon the Boltzmann equation in the full domain of the velocity variable, v.

[1] L. S. KOTHARI, *Nucl. Sci. Eng.*, this issue, p. 402.

It is not at all difficult to find a qualitative explanation for this phenomenon; indeed, Dr. Michael and I convinced ourselves of one in 1962, when we first discussed the bounds on the discrete λ's. It is this: When the system is small enough, no discrete λ's will exist, and the evolution of the pulse will be described in terms of a continuous spectrum of decay constants. Then, the amplitude, $A(\lambda)$, which is associated with the λ's between λ and $\lambda + d\lambda$, will play a particularly important role. $A(\lambda)$ will reflect the scattering properties of the moderator; in the case of a coherent crystalline sample, it will show considerable oscillation, while it will vary smoothly when the moderator is an incoherent scatterer. Since a sharp peak (or valley) in $A(\lambda)$ at $\lambda = \lambda_P > \lambda^*$ produces an effect upon integration, rather like that of a discrete mode $\sim \exp(-\lambda_P t)$, one sees that such a pseudo-mode may well be found in a coherent scatterer. Further, we shall see that the value of λ_P' one obtains lies close to that suggested earlier by de Saussure[4]. Of course, λ_P, while it may dominate the decay of the pulse, is in no way connected with a fundamental or asymptotic mode. After a sufficiently long time, the portion of the continuous spectrum in the neighborhood of λ^* will dominate the decay.

I can make the argument more quantitative by treating the leakage of neutrons by diffusion theory. (The diffusion approximation is hardly justified, but it yields the main features of the argument.) Then, for sufficiently large buckling,

$$N(v,t) = \int_{\lambda^*}^{\infty} d\lambda e^{-\lambda t} A(\lambda, v). \qquad (1)$$

One can show, now, that[2]

$$A(\lambda,v) = \left[P\frac{1}{(v\Sigma-\lambda)} + f(\lambda)\,\delta(v\Sigma-\lambda)\right] g(\lambda,v), \qquad (2)$$

where P denotes 'principal value,' f and g are 'smooth' in λ, and

$$v\Sigma \equiv v\Sigma_{s,\mathrm{inel}} + vD(v)B^2. \qquad (3)$$

The response, $R(t)$, of a $1/v$ detector to the pulse will be given by the integral of Eq. (1) with respect to v Equation (2) tells us that the result is

$$\int_0^{\infty} dv N(v,t) = \int_{\lambda^*}^{\infty} d\lambda\, e^{-\lambda t}\left[B(\lambda) + \frac{f(\lambda)g(\lambda,v(\lambda))}{\left|\frac{d}{dv}\, v\Sigma\right|_{v(\lambda)}}\right] \qquad (4)$$

In Eq. (4) $v(\lambda)$ is the solution to $v\Sigma(v) = \lambda$, an equation assumed, for simplicity, to have only one solution. The quantity in square brackets is the amplitude, $A(\lambda)$, mentioned above. Its fluctuating nature stems from the denominator of the second term. When coherent scattering

[2] See, for example, R. BEDNARZ and J. MIKA, *J. Math. Phys.*, **4**, 1285 (1964).

[3] These conclusions are illustrated by the recent numerical calculations of A. GHATAK and H. C. HONECK, *Nucl. Sci. Eng.*, **21**, 227 (1965); *J. Nucl. Eng.*, **19**, 1 (1965).

[4] G. de SAUSSURE, *Nucl. Sci. Eng.*, **12**, 433 (1962).

causes $D(v)$ to fluctuate, the derivative of $v\Sigma$ fluctuates even more sharply, and the effect upon $R(t)$ is to make it appear to be composed of discrete, exponential modes. This is the effect we seek. It should account for the experimental results without recourse to a cut-off in velocity[3].

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On the Use of the Poincaré-Bertrand Formula in Neutron Transport Theory

In a recent letter, Jacobs and McInerney[1] have questioned some of the results obtained by the normal-mode method[2] in one-speed neutron transport theory. For instance, the version of the full-range closure relation (for isotropic scattering), which is implicit in some previously reported results[2,3],

$$\frac{\phi(L,\mu)\phi(L,\mu')}{M_+} + \frac{\phi(-L,\mu)\phi(-L,\mu')}{M_-} + \int_{-1}^{1} \frac{\phi(\nu,\mu)\phi(\nu,\mu')}{M(\nu)} d\nu = \frac{\delta(\mu - \mu')}{\mu}, \quad (1A)$$

is criticized. Instead, the right-hand side should read as[1]

$$\frac{\lambda^2(\mu)}{M(\mu)} \delta(\mu - \mu'), \quad (1B)$$

in Mika's notation[3]. This criticism also applies to a number of previously established results for Green's function and albedo problems, where integrals similar to that in Eq. (1) appear in the expressions for the angular density.

The difference between (1A) and (1B) lies in the interpretation of Cauchy principal-value integrals, if the integrand has two singularities that are allowed to merge. Such integrals are handled by the Poincaré-Bertrand formula[4],

$$\int d\nu \int d\mu' F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} = \int d\mu' \int d\nu F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} + \pi^2 F(\mu,\mu), \quad (2B)$$

with μ inside the interval over which both integrations are carried out.

This formula is not completely clear until we define what is meant by the integral over ν on the right-hand side when $\mu' \to \mu$. This is done by using the identity[4]

$$P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} \equiv \frac{1}{\mu - \mu'}\left[P\frac{1}{\nu - \mu} - P\frac{1}{\nu - \mu'}\right], \quad (3B)$$

with the agreement that the limit $\mu' \to \mu$ may be carried out only after integration over ν.

Other definitions of the limit of that integral can be proposed that lead to an infinity like $\delta(\mu - \mu')$. Since there is some freedom in the choice of the definition, we take the liberty to modify Eq. (3B) in such a way that the extra term from the Poincaré-Bertrand formula is incorporated here. That is, we define[5]

$$P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} \equiv \frac{1}{\mu - \mu'}\left[P\frac{1}{\nu - \mu} - P\frac{1}{\nu - \mu'}\right] + \pi^2\delta(\nu - \mu)\delta(\nu - \mu'), \quad (3A)$$

so that (2B) is replaced by

$$\int d\nu \int d\mu' F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} = \int d\mu' \int d\nu F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'}. \quad (2A)$$

As in version B, each side of Eq. (3A) applies to the corresponding side of Eq. (2A). That is, the left-hand side of Eq. (3A) can be used only if the integration over μ or μ' comes first, whereas we use the right-hand side if the integration over ν is to be carried out first.

To summarize, we now have two versions of the Poincaré-Bertrand formula: Eqs. (2B) and (3B) or, alternatively, (2A) and (3A). With either version, a consistent system of formulas for neutron transport theory can be constructed. Jacobs and McInerney have demonstrated this for version B, and several earlier authors for version A. For example, in the two versions the integrand occurring in Eq. (1) is analyzed according to the following identities:

$$\phi(\nu,\mu)\phi(\nu,\mu') \equiv \frac{c\nu}{2}\frac{1}{\mu - \mu'}\left[\phi(\nu,\mu) - \phi(\nu,\mu')\right] + \left[\lambda^2(\mu) + \left(\frac{1}{2}\pi c\mu\right)^2\right]\delta(\nu - \mu)\delta(\nu - \mu'), \quad (4A)$$

$$\phi(\nu,\mu)\phi(\nu,\mu') \equiv \frac{c\nu}{2}\frac{1}{\mu - \mu'}\left[\phi(\nu,\mu) - \phi(\nu,\mu')\right] + \lambda^2(\mu)\delta(\nu - \mu)\delta(\nu - \mu'). \quad (4B)$$

This explains the difference between Eqs. (1A) and (1B).

For neutron transport theory, version A is to be recommended for two reasons. The first is tradition; except for the work of Jacobs and McInerney[1,6], version A has been used consistently in this field, although sometimes without due explanation. Secondly, many formulas and derivations are much simpler and shorter in this version because Eq. (2A) permits us to formally switch orders of integration.

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[1] A. M. JACOBS and J. J. McINERNEY, *Nucl. Sci. Eng.*, **22**, 119-120 (1965).
[2] K. M. CASE, *Ann. Phys.*, **9**, 1-23 (1960).
[3] J. MIKA, *Nucl. Sci. Eng.*, **11**, 415-427 (1961).
[4] N. I. MUSKHELISHVILI, *Singular Integral Equations*, Noordhoff, Groningen (1953).
[5] I. KUŠČER, N. J. McCORMICK and G. C. SUMMERFIELD, *Ann. Phys.*, **30**, 411-421 (1964).
[6] J. J. McINERNEY, *Nucl. Sci. Eng.*, **22**, 215-234 (1965).

A Note on the Adjoint Function in the Time Optimal Xenon Shutdown Problem

Smith and Roberts[1] (hereinafter I) have recently applied the Pontryagin theorem to time optimal xenon shutdown in

[1] J. J. ROBERTS and H. P. SMITH, Jr., "Time Optimal Solution to the Reactivity-Xenon Shutdown Problem," *Nucl. Sci. Eng.*, **22**, 470 (1965).