a smaller chance of crossing the void gap and entering the adjacent non-void regions.

Therefore, one should undertake some numerical checks of the method described here before applying it to specific problems. I do not have an opportunity to do this, at present, and have submitted this letter trusting someone else will.

Tom Olsen

Martin Company, Nuclear Division
Martin Company
Nuclear Division

Received January 9, 1964
Revised September 28, 1964

Comments on "Generalizations of Fick's Law"*

In this letter we present several comments on a letter by Kostin[1]. For brevity, we shall refer to "Eq. (n) of Ref. 1" as "(K-n)."

(I) The "generalized Fick's law" (K-11) is alleged to have the simplicity of Fick's law. We believe that this is not so, for the following reasons: 1) The quantity which appears under the derivative sign is $\overline{v^2}(z)N(z)$ and not $N(z)$. 2) The $\overline{k}_{tr}(z)$, which plays the role of a reciprocal diffusion coefficient, is not an intensive property of the medium[2,3]. Even in a homogeneous medium, $\overline{k}_{tr}(z)$ may vary with z. 3) the $\overline{k}_{tr}(z)$ depends explicitly on the neutron distribution $n(z,v,\underset{\sim}{\omega})$. In the time-dependent form (K-7), the $\overline{k}_{tr}(z,t)$ depends explicitly on $n(z,v,\underset{\sim}{\omega},t)$. It follows that $\overline{k}_{tr}(z,t)$ may be a function of time in a system with fixed composition and fixed boundaries.

(II) The definition (K-5) of $\alpha(z,v')$ is, with trivial changes in notation,

$$\int \Sigma_s(z,v'\to v, \underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})v\,dv\,d\omega = v'\alpha(z,v')\Sigma_s(z,v').$$

This definition does not lead to (K-7) or (K-11); however a corrected definition

$$\int \Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\,v\mu\,dv\,d\omega = v'\mu'\alpha(z,v')\Sigma_s(z,v') \tag{1}$$

does lead to (K-7) and (K-11). (In an appendix at the close of this letter, we answer a question regarding the definition of α in Eq. (1).) To verify this statement, apply the operator $\int \mu v\,dv\,d\omega$ to (K-4) to obtain

$$\begin{aligned}\frac{\partial J(z,t)}{\partial t} = &-\frac{\partial}{\partial z}\int \mu^2 v^2 n(z,v,\underset{\sim}{\omega},t)\,dv\,d\omega\, - \\ &-\int \mu v^2\,\Sigma(z,v)\,n(z,v,\underset{\sim}{\omega},t)\,dv\,d\omega\, + \\ &+\int \mu v v'\,\Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega}) \times \\ &\times n(z,v',\underset{\sim}{\omega}',t)\,dv'\,d\omega'\,dv\,d\omega\ .\end{aligned} \tag{2}$$

By use of the definition of $\alpha(z,v')$ from Eq. (1) we may write the final term as

$$\int \mu'(v')^2\,\alpha(z,v')\,\Sigma_s(z,v')\,n(z,v',\underset{\sim}{\omega}',t)\,dv'\,d\omega'$$

$$= \int \mu v^2\,\alpha(z,v)\,\Sigma_s(z,v)\,n(z,v,\underset{\sim}{\omega},t)\,dv\,d\omega$$

and obtain (K-7) and (K-11) unaltered. The definition (K-5) does not allow this transformation of the final term of Eq. (2).

(III) If the goal is to remove the one-velocity restriction from (K-1), then we feel that a better way is to apply the operator $\int \mu\,d\omega$ rather than $\int v\mu\,dv\,d\omega$ to (K-4). Furthermore, we feel that it is preferable to include on the right-hand side of (K-4) a source term $S(z,v,\underset{\sim}{\omega},t)$. This source may be caused by fissions, and we assume it to be isotropic in the sense that $\int \mu S\,d\omega \equiv 0$. In the time-independent case, the immediate result of applying $\int \mu\,d\omega$ to this modified (K-4) is

$$\begin{aligned}0 = &-v\frac{\partial}{\partial z}\int \mu^2 n(z,v,\underset{\sim}{\omega})\,d\omega - v\Sigma(z,v)\int \mu n(z,v,\underset{\sim}{\omega})\,d\omega\, + \\ &+\int \mu v' n(z,v',\underset{\sim}{\omega}')\Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\,dv'\,d\omega'\,d\omega.\end{aligned}$$

With the definitions

$$\begin{aligned}J(z,v) &= v\int \mu n(z,v,\underset{\sim}{\omega})\,d\omega\ ,\\ N(z,v) &= \int n(z,v,\underset{\sim}{\omega})\,d\omega\ ,\\ \Sigma_s(z,v'\to v) &= \int \Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\,d\omega\ ,\\ \int \mu\Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\,d\omega &= \mu'\overline{\mu_0}(z,v'\to v)\Sigma_s(z,v'\to v),\end{aligned} \tag{3}$$

and

$$\overline{\mu^2}(z,v) = \int \mu^2 n(z,v,\underset{\sim}{\omega})\,d\omega/N(z,v)\ ,$$

this result becomes

$$\begin{aligned}0 = &-v\frac{\partial}{\partial z}\left[\overline{\mu^2}(z,v)N(z,v)\right] - \Sigma(z,v)J(z,v)\, + \\ &+\int J(z,v')\overline{\mu_0}(z,v'\to v)\,\Sigma_s(z,v'\to v)\,dv'\ .\end{aligned} \tag{4}$$

(In an appendix at the close of this letter, we give concise answers to certain questions regarding the definition of $\overline{\mu_0}$ in Eq. (3).)

---

*Work performed under the auspices of the USAEC.

[1]M. D. KOSTIN, *Nucl. Sci. Eng.* **19**, 252-254 (1964).
[2]A. M. WEINBERG and E. P. WIGNER, *The Physical Theory of Neutron Chain Reactors*, p. 413, Univ. of Chicago Press, Chicago, (1958).
[3]E. A. GUGGENHEIM, in *Encyclopedia of Physics* (S. Flügge, ed.), Vol. III/2, p. 3, Springer-Verlag, Berlin, (1959).

To obtain a "generalized Fick's law", we define

$$\Sigma_{tr}(z,v) = \Sigma(z,v) - \int J(z,v')\overline{\mu_0}(z,v'\to v) \times$$
$$\times \Sigma_s(z,v'\to v)dv'/J(z,v)$$

and write

$$J(z,v) = -\frac{v}{\Sigma_{tr}(z,v)}\frac{\partial}{\partial z}[\overline{\mu^2}(z,v)N(z,v)]. \quad (5)$$

Equation (5) does not have the simplicity of Fick's law for the reasons already listed in comment (I). In terms of the quantities defined above, Eq. (1) becomes

$$\int v\mu'\overline{\mu_0}(z,v'\to v)\Sigma_s(z,v'\to v)dv = v'\mu'\alpha(z,v')\Sigma_s(z,v') . \quad (6)$$

Consequently (K-11) may be derived from Eq. (4) by application of the operator $\int vdv$. Since information is lost in application of the operator $\int vdv$, Eq. (4) cannot be derived from (K-11). If we assume that scattering takes place without change in speed, then we may write

$$\int \mu\Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})d\omega = \int\mu\delta(v'-v) \times$$
$$\times \Sigma_s(z,v',\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})d\omega = \mu'\delta(v'-v)\overline{\mu_0}(z,v')\Sigma_s(z,v') .$$

It follows that for scattering without change in speed, (K-1), (K-2) and (K-3) are valid with J, Σ_{tr}, $\overline{\mu^2}$, N, Σ, $\overline{\mu_0}$ and Σ_s all functions of z *and* v. The v dependence is meaningful because of the v dependence of the fission source term. Here $\Sigma_{tr}(z,v)$ is an intensive property of the medium.

(IV) Instead of writing and using a 'generalized Fick's law' in the form of (K-11) or of Eq. (5), we believe that it is more direct to use the first velocity moment Eq. (2) or the first angular moment Eq. (4). Each of these first-moment equations is equivalent to a corresponding 'generalized Fick's law'. Usually a first-moment equation (or a 'generalized Fick's law') is not sufficient by itself for solution of problems in reactor analysis, so additional equations are needed. An appropriate zeroth-moment equation, analogous to (K-15), is frequently employed. Neither these moment equations nor their application in reactor analysis nor the idea of neutron pressure is new. Persiani[4] derives these moments of the Boltzmann equation in three-dimensional form. He also gives application in reactor analysis and discusses the physical significance of these and of higher moments in terms of neutron pressure and other quantities. For a review and further application of methods of angular and velocity moments in transport theory, see Ziering and Kahalas[5] and Johnston[6].

(V) The part of Ref. 1 from (K-1) through (K-3) and from (K-13) through (K-20), including Table I, is essentially independent of the remainder of the reference. The use of (K-11) in the two examples is clearly not essential. Table I, giving $\overline{\mu^2}(z)$ as a function of $z\Sigma_s$, is instructive.

(VI) Weinberg and Noderer[7] and Amaldi[8] present helpful and appropriate discussions of forms of Fick's law.

APPENDIX

Here we give concise answers to the following questions regarding the definitions of α and $\overline{\mu_0}$ in Eqs. (1) and (3): 1) are α and $\overline{\mu_0}$ independent of $\underset{\sim}{\omega}'$ and $\underset{\sim}{\omega}$? 2) Is the $\overline{\mu_0}$ of Eq. (3) the same as the more conventional $\overline{\mu_0}$ defined by

$$\frac{\int \underset{\sim}{\omega}'\cdot\underset{\sim}{\omega}\,\Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})d\omega}{\int \Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})d\omega} = \frac{2\pi\int_{-1}^{+1}\mu_0\Sigma_s(z,v'\to v,\mu_0)d\mu_0}{2\pi\int_{-1}^{+1}\Sigma_s(z,v'\to v,\mu_0)d\mu_0} \quad ?$$

Now by the Fenyö-Khekke theorem[9]

$$\int\mu\Sigma_s(z,v'\to v,\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})d\omega = 2\pi\mu'\int_{-1}^{+1}\mu_0\Sigma_s(z,v'\to v,\mu_0)d\mu_0 .$$

It follows that the answers to the questions about $\overline{\mu_0}$ are affirmative. Since $\overline{\mu_0}$ is independent of $\underset{\sim}{\omega}'$ and $\underset{\sim}{\omega}$, the relation between α and $\overline{\mu_0}$ in Eq. (6) implies that α is independent of $\underset{\sim}{\omega}'$ and $\underset{\sim}{\omega}$.

A. W. Solbrig, Jr.

Phillips Petroleum Company
Atomic Energy Division
Idaho Falls, Idaho

Received July 21, 1964

[4]P. J. PERSIANI, "Special Lectures on The Physical Foundation of Reactor Analysis," ANL-6227, Argonne National Laboratory, (1960).

[5]S. ZIERING and S. L. KAHALAS, in *Magnetohydrodynamics* (A. B. Cambel, T. P. Anderson and M. M. Slawsky, eds.), pp. 33-54, Northwestern Univ. Press, Evanston, (1962).

[6]T. W. JOHNSTON, *Phys. Rev.* **120**, 1103-1111 (1960).

[7]A. M. WEINBERG and L. C. NODERER, "Theory of Neutron Chain Reactions," Vol. I, pp. 5-25, CF-51-5-98 (AECD-3471), Oak Ridge National Laboratory (1951).

[8]E. AMALDI, in *Encyclopedia of Physics* (S. Flügge, ed.), Vol. XXXVIII/2, pp. 520-527, Springer-Verlag, Berlin, (1959).

[9]V. S. VLADIMIROV, "Mathematical Problems in the One-Velocity Theory of Particle Transport," AECL-1661, pp. 187 and 288, Atomic Energy of Canada Limited, Chalk River, Ontario (1963).