or

$$\sum_{\ell=0}^{n} (2\ell + 1) P_\ell(\mu') P_\ell(\mu) = (n+1) \left[\frac{P_{n+1}(\mu') P_n(\mu) - P_n(\mu') P_{n+1}(\mu)}{\mu' - \mu} \right], \tag{A7}$$

which is Eq. 7.

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Some Tests of Coveyou's Anisotropic Selection Technique*

The application of R. R. Coveyou's anisotropic selection technique[1] and its proper inclusion into the Monte Carlo neutron-transport code[2] 05R have been tested by several calculations and comparisons. The technique is the result of its originator's initial attempt to devise a scheme for choosing from an anisotropic distribution, and so there was no older, tried technique against which either its accuracy or its effect on machine time could be tested for arbitrary angular distributions. Problems for which analytic solutions exist offer a better test of the accuracy of the technique than do problems solved by another Monte Carlo method which is itself subject to error in any case. For this reason one-velocity problems with isotropic scattering in the laboratory system were chosen for study with the anisotropy being introduced in the center-of-mass system in such a way as to result in isotropic laboratory scattering.

Beach *et al.*[3] have solved various one-velocity neutron-transport problems with isotropic scattering in the laboratory system by semianalytical methods. The problem chosen for comparison was the calculation of the flux from a plane isotropic source in a medium having a scattering cross section equal to half its total cross section. So that all parts of the code could be tested, the 05R calculations were made with constant cross sections, but with neutron slowing down permitted. The fluxes were obtained by a simple statistical estimation procedure which extended the path of a neutron from each collision point to the various planes at which the flux was desired, the contribution from each collision being given by

$$\frac{W}{|\mu|} e^{-|(x'-x)|/\lambda|\mu|}, \tag{1}$$

where

W = the statistical weight of the neutron after collision
x' = the x coordinate of the plane at which the flux is to be estimated
x = the x coordinate of the collision point
λ = the total mean free path
μ = the cosine of the angle between the neutron velocity vector and the x axis

and the source is in the y-z plane at the origin.

In order to isolate the systematic errors inherent in the Monte Carlo technique, the 05R calculations were made both for a medium whose scattering was isotropic in the center of mass and whose mass was 240, thus making the scattering in the laboratory system very nearly isotropic, and for a medium whose scattering was anisotropic in the center of mass. The latter medium was a half-and-half mixture of scatterers having masses of 2 and 3, each with a P_8 approximation to the center-of-mass scattering angular distribution which yields an isotropic-laboratory distribution. The only reason for using a mixture rather than a single scatterer was to test the code's ability to handle mixtures of anisotropic scatterers. Two thousand neutrons were run for each case.

The ℓ-th Legendre coefficient in an expansion of the center-of-mass angular distribution for a scatterer of mass A which gives an isotropic laboratory distribution is given by $(\ell+1)/(2\ell+1)(-1/A)^\ell$: The center-of-mass angular distribution resulting from a P_8 approximation for a mass 2 scatterer is given in Fig. 1, where the probability per unit cosine $F(\mu)$ is plotted as a function of the cosine of the center-of-mass scattering angle μ. Also shown as vertical bars are the ϕ_k's for the P_8 approximation required by Coveyou's method. They occur at the roots of P_9. It is somewhat startling to see the probability of selecting $\mu = 0.968$ fall below that of selecting $\mu = -0.836$ while $F(\mu)$ is rising. This may be qualitatively seen by interpreting the ϕ_k's as the integral of $F(\mu)$ over some appropriate interval including the k-th root. As the roots

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[1] R. R. COVEYOU, "A Monte Carlo Technique for Selecting Neutron Scattering Angles from Anisotropic Angular Distributions," *Nucl. Sci. Eng.* this issue, p. xxx.

[2] R. R. COVEYOU, J. G. SULLIVAN and H. P. CARTER, *Codes for Reactor Computations*, p. 267, International Atomic Energy Agency, Vienna, (1961).

[3] L. A. BEACH *et al.*, "Comparison of Solutions to the One-Velocity Neutron Diffusion Problem," NRL-5052 (Dec. 23, 1957).

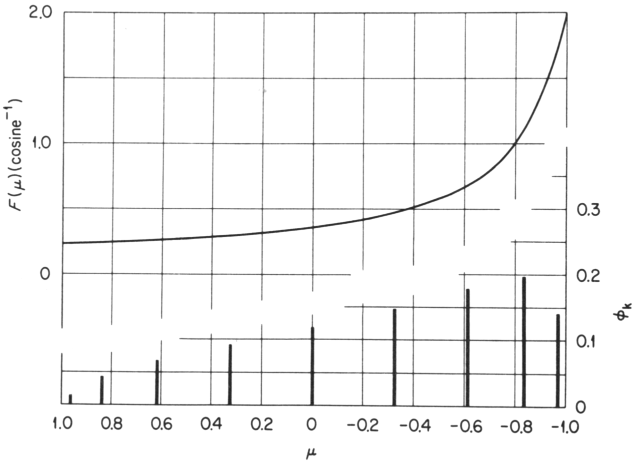

Fig. 1. A P_8 approximation to the distribution function $F(\mu)$ for the cosine of the center-of-mass scattering angle μ to give isotropic laboratory scattering for a mass 2 scatterer and the ϕ_k required by Coveyou's selection technique.

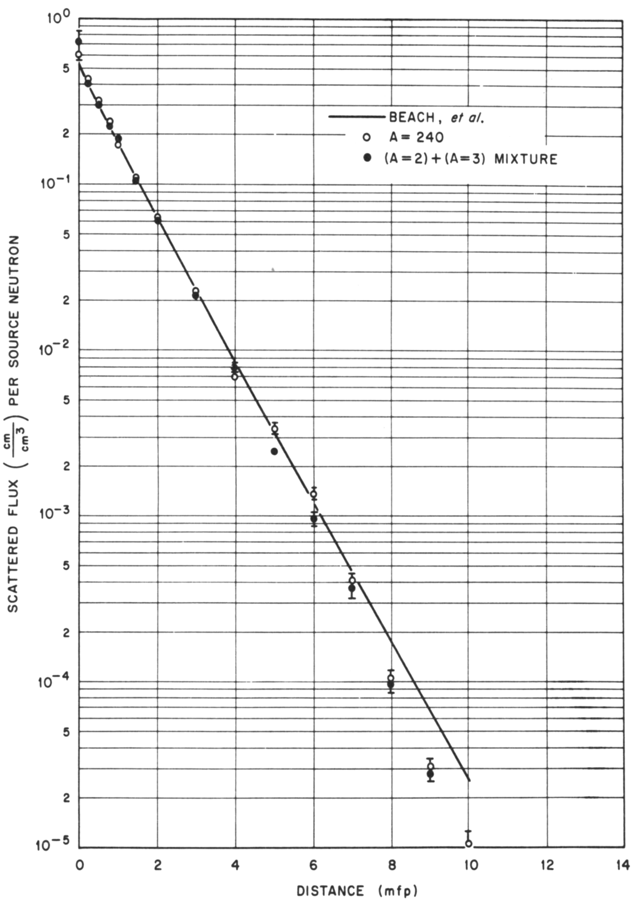

Fig. 2. A comparison of one-velocity scattered fluxes due to plane isotropic source in an isotropic scattering medium (Σ_s/Σ_t = 0.5) computed by semianalytic and Monte Carlo methods as a function of distance from the source. The curve is the semianalytic result; the points are Monte Carlo results.

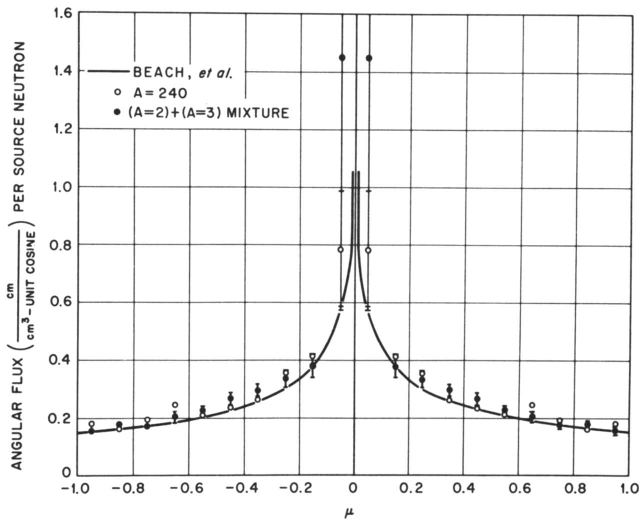

Fig. 3. A comparison of the angular distributions of scattered neutrons at zero mean free path from the source plane as computed by semianalytic and Monte Carlo methods. The curve is the semianalytic result; the points are Monte Carlo results.

crowd closer together as $|\mu|$ increases, the intervals between the roots decrease so that the integrals of $F(\mu)$, the ϕ_k's, may decrease even though $F(\mu)$ itself is increasing. The ϕ_k's for a uniform distribution will peak near or at μ = 0, depending on whether the order of approximation is odd or even, and fall off symmetrically as $|\mu|$ approaches unity. The values of the cosines of the center-of-mass scattering angles, the angles themselves, and the ϕ_k's for mass 2 and mass 3 are given in Table I.

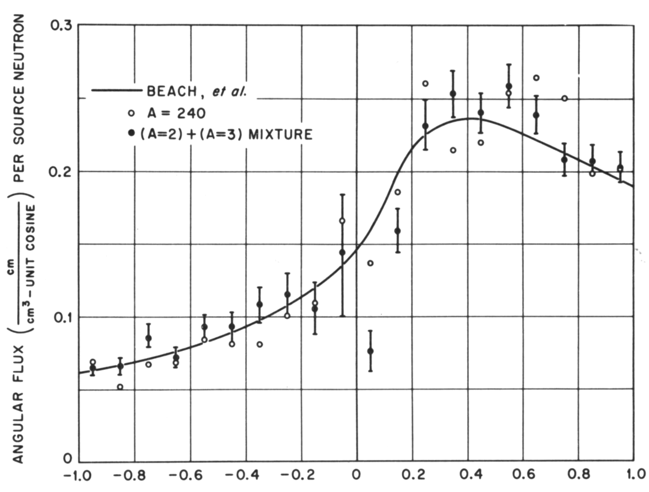

Fig. 4. A comparison of the angular distributions of scattered neutrons at 0.5 mean free path from the source plane as computed by semianalytic and Monte Carlo methods. The curve is the semianalytic result; the points are Monte Carlo results.

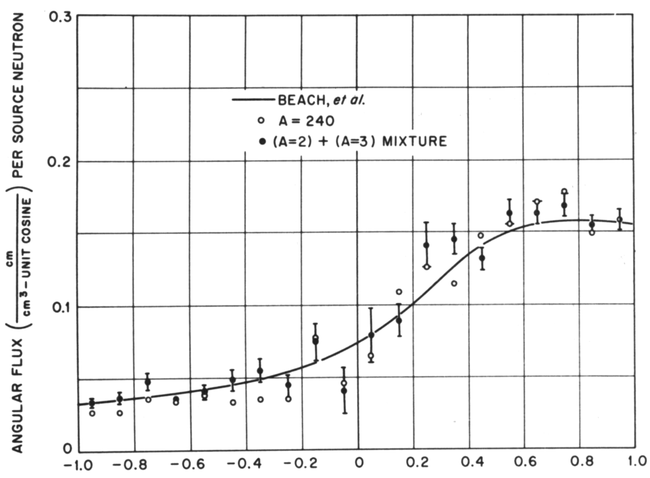

Fig. 5. A comparison of the angular distributions of scattered neutrons at 1.0 mean free path from the source plane as computed by semianalytic and Monte Carlo methods. The curve is the semianalytic result; the points are Monte Carlo results.

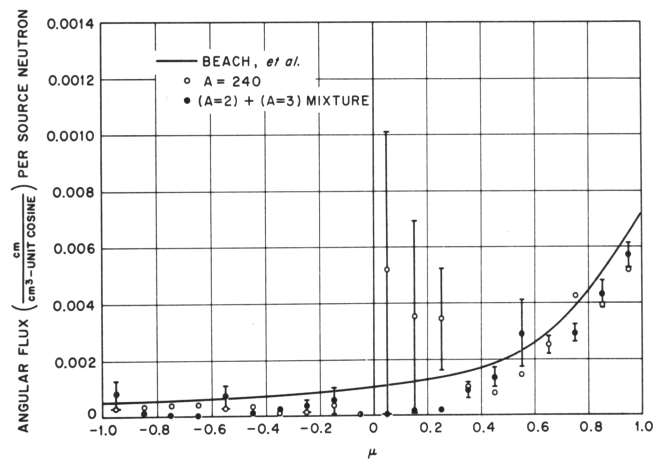

Fig. 6. A comparison of the angular distributions of scattered neutrons at 5.0 mean free paths from the source plane as computed by semianalytic and Monte Carlo methods. The curve is the semianalytic result; the points are Monte Carlo results.

TABLE I

The Cosine of the Center-of-Mass Scattering Angle, the Center-of-Mass Scattering Angle, and ϕ_k for Mass 2 and Mass 3 Scatterers

Cosine of the Center-of-Mass Scattering Angle	Center-of-Mass Scattering Angle (deg)	ϕ_k (Mass 2)	ϕ_k (Mass 3)
0.96816	14.497	0.01828	0.02309
0.83603	33.277	0.04243	0.05359
0.61337	52.166	0.06710	0.08375
0.32425	71.080	0.09164	0.11317
0.	90.000	0.11849	0.14099
-0.32425	108.920	0.14648	0.16452
-0.61337	127.834	0.17834	0.17618
-0.83603	146.723	0.19692	0.15810
-0.96816	165.503	0.14032	0.08661

TABLE II

Comparison of the Multiplication Constants Computed by Various 05R Systems with Exact Results for Infinite Slabs, Infinite Cylinders, and Spheres

	Multiplication Constant[a]		
	Infinite Slab (r = 2.1134 mfp)[b]	Infinite Cylinder (r = 3.5783 mfp)	Sphere (r = 4.8727 mfp)
05R: One-velocity; isotropic	0.979 ± 0.008	1.00 ± 0.01	0.991 ± 0.008
05R: Fission spectrum; A = 15 000	0.992 ± 0.009	0.980 ± 0.014	0.999 ± 0.013
05R: Fission spectrum; (A = 2) + (A = 3)	0.994 ± 0.010	1.010 ± 0.010	1.009 ± 0.008
Exact	1.00	1.00	1.00

[a] The number of secondaries per collision was arbitrarily taken as 1.1.
[b] r = half thickness.

The fluxes calculated by the 05R code are compared, as a function of distance, in Fig. 2 with the scattered flux computed by Beach *et al.* One notes first that the 05R results are consistent with each other. The low 05R fluxes at distances greater than 6 mfp (mean free paths) demonstrate the characteristic Monte Carlo systematic error mentioned above and are caused by an insufficient number of neutrons penetrating to distances far from the source. The high values of 05R results compared with those of Beach *et al.* at the origin are no doubt due to the contribution of a few neutrons having very small values of μ in Eq. 1.

The angular distributions of scattered neutrons at 0, 0.5, 1, and 5 mfp from the source plane are compared with the results of Beach *et al.* in Figs. 3-6, respectively. The 05R results are averages over a 0.1 interval in μ, plotted at the midpoint of the interval. Noticeable in the data for the mixture of light scatterers is the large contribution in the 0 to 0.1 interval adjacent to the source plane due to the contributions of low-μ neutrons. The semianalytic result, of course, goes to infinity at the source plane. In all cases of 05R results are consistent. Agreement with the semianalytic results is good to 1 mfp, and not really poor even at 5 mfp. At this depth, of course, the agreement is considerably better for positive μ than for negative μ, since so few neutrons penetrate deeply into the medium and are scattered back toward the origin.

The error bars shown on some of the points are consistent for both media and represent the standard deviation computed from the relation

$$\sigma = \sqrt{\frac{\frac{1}{N}\sum_{1}^{N} W^2 - \left(\frac{1}{N}\sum_{1}^{N} W\right)^2}{N-1}} \quad ,$$

where W = the neutron weight and N = the total number of source neutrons.

A second test of the anisotropic-scattering selection technique was the calculation of the multiplication constant for one-velocity neutrons in infinite slabs, infinite cylinders, and spheres of media having isotropic scattering in the laboratory system. Exact results for such problems have been tabulated by Carlson and Bell[4]. The number of secondaries per collision was arbitrarily taken as 1.1 for the comparison.

Three calculations were performed for each of the configurations. The first was a strictly one-velocity problem with isotropic scattering in the laboratory system. This was intended to evaluate the accuracy of the Monte Carlo method in computing a multiplication constant. The second considered a scattering medium having a mass of 15 000 with isotropic scattering in the center-of-mass system but introduced the neutrons in a fission spectrum. This calculation, by comparison with the first, tested the equivalence of the constant cross-section one-velocity and multi-velocity cases on 05R. The third calculation was the test of the anisotropic scattering treatment. The medium was a half-and-half mixture of scatterers having masses of 2 and 3, with each having a P_8 approximation to the center-of-mass distribution which yielded an isotropic angular distribution in the laboratory system. At least 20 iterations of 400 histories each were performed in each calculation, with the first five being discarded in the computation of the multiplication constant to permit spatial convergence of the source distribution.

The results of the calculations are compared with the exact results in Table II. The calculations testing the anisotropic-scattering selection technique, those for the $(A = 2) + (A = 3)$ mixture, are consistent with the other 05R results and are in agreement with the exact results.

The consistency of the anisotropic-scattering medium results obtained with the Monte Carlo calculations and the demonstrated agreement with semianalytic and exact results indicate that Coveyou's selection technique is appropriate and has been properly incorporated into the 05R code.

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[4]B. G. CARLSON and G. I. BELL, "Solution of the Transport Equation," P/2386, *Proc. U. N. Intern. Conf. Peaceful Uses Atomic Energy, 2nd,* Geneva, (1958).

The Albedo Problem

In a recent note, Rafalski[1] has considered the problem of computing the probability that a neutron will be reflected (the albedo) if it is perpendicularly incident on a semi-infinite half-space. His method of solution consisted of introducing an approximation into the integral transport equation describing the problem and led to a simple analytic result for the albedo. We show that the application of the variational method to this problem also leads to a simple analytic expression for the albedo and that this expression is significantly more accurate. Exact formulations

[1]P. RAFALSKI, *Nucl. Sci. Eng.*, **19**, 378 (1964).