in graphite has been measured or inferred. It was my intent in a letter to the editor[1] to demonstrate that the pulsed-neutron method and the poison method yield consistent values, as they should, since the $(\lambda, B_g{}^2)$ space in the pulsed-neutron experiment can be considered as an extension of the $(\Sigma_a, 1/L^2)$ space in the poison experiment. This agreement is to be expected because both methods rely primarily upon diffusion theory, with similar assumptions, inferences and restrictions applying to both cases. I assumed λ_t to be the Maxwellian average of the transport mean free path in the infinite medium, while the CB^4 term corrects for deviations from Maxwellian spectrum in the finite medium.

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[1]G. A. PRICE, "A Note on the Measurement of the Transport Mean Free Path of Thermal Neutrons in Graphite by a Poison Method," *Nucl. Sci. Eng.*, 18, pp. 400-413, (1964).

A Monte Carlo Technique for Selecting Neutron Scattering Angles from Anisotropic Distributions*

The angular distributions of elastically scattered neutrons become anisotropic in the center-of-mass system at neutron energies above about 100 keV, the degree of anisotropy increasing with neutron energy. For many problems in neutron transport at such energies, a satisfactory solution is obtained by taking account of the anisotropy is only an approximate fashion. Diffusion theory, for example, accounts approximately for anisotropy by the use of the transport cross section, $\Sigma - \bar{\mu}\,\Sigma_s$, where Σ is the total cross section, Σ_s is the elastic scattering cross section, and $\bar{\mu}$ is the average cosine of the laboratory scattering angle. For some problems, however, a more accurate treatment of anisotropic scattering is required. Calculations of neutron distributions deep within thick shields or, at the other extreme, in very small critical systems are examples of such problems.

When the Monte Carlo method is used to solve neutron-transport problems, angular distributions can be included to as high a degree of accuracy as desired; however, in general, the higher the degree of accuracy demanded the more costly the solution, since the selection of a scattering angle becomes an elaborate procedure and requires a large amount of computer time. A selection method is presented here which gives the same accuracy as that obtained by a straightforward selection from a Legendre expansion but requires considerably less computer time.

Express the distribution function, $F(\mu)$, of the cosine of the scattering angle, μ, as the Legendre series

$$F(\mu) = \sum_{\ell=0}^{n} \frac{2\ell+1}{2} f_\ell P_\ell(\mu) + \sum_{\ell=n+1}^{\infty} \frac{2\ell+1}{2} f_\ell P_\ell(\mu), \tag{1}$$

where

$$f_\ell = \int_{-1}^{1} F(\mu)\, P_\ell(\mu)\, d\mu, \tag{2}$$

but only the first sum is known, while the sum of the remainder of the terms from $n+1$ to infinity is unknown. Consider a second distribution function $G(\mu)$ given by

$$G(\mu) = \sum_{k=0}^{n} \phi_k\ \delta(\mu - \theta_k) \quad . \tag{3}$$

Expanding the δ-function gives

$$\begin{aligned} G(\mu) &= \sum_{k=0}^{n} \phi_k \left\{ \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(\theta_k)\, P_\ell(\mu) \right\} \\ &= \sum_{\ell=0}^{n} \frac{2\ell+1}{2} \left\{ \sum_{k=0}^{n} \phi_k P_\ell(\theta_k) \right\} P_\ell(\mu) + \\ &\quad + \sum_{\ell=n+1}^{\infty} \frac{2\ell+1}{2} \left\{ \sum_{k=0}^{n} \phi_k P_\ell(\theta_k) \right\} P_\ell(\mu) \, . \end{aligned} \tag{4}$$

Now if one sets

$$f_\ell = \sum_{k=0}^{n} \phi_k P_\ell(\theta_k)\ , \tag{5}$$

Eq. 4 and Eq. 1 are identical; hence they are good to the same order of approximation when both are truncated at n.

To find the ϕ_k, multiply Eq. 5 by $\frac{2\ell+1}{2} P_\ell(\theta_j)$ and sum over ℓ from 0 to n

$$\sum_{k=0}^{n} \phi_k \sum_{\ell=0}^{n} \frac{2\ell+1}{2} P_\ell(\theta_k) P_\ell(\theta_j) = \sum_{\ell=0}^{n} \frac{2\ell+1}{2} f_\ell P_\ell(\theta_j). \tag{6}$$

As shown in Appendix A,

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$$\sum_{\ell=0}^{n} \frac{2\ell+1}{2} P_\ell(\mu') P_\ell(\mu) =$$

$$= \frac{n+1}{2} \left[\frac{P_{n+1}(\mu') P_n(\mu) = P_n(\mu') P_{n+1}(\mu)}{\mu' - \mu} \right] . \tag{7}$$

Now let the $n+1$ values of θ_k be the roots of $P_{n+1}(\mu)$; i.e.

$$P_{n+1}(\theta_k) = 0 \quad . \tag{8}$$

Then, from Eq. 7,

$$\sum_{\ell=0}^{n} \frac{2\ell+1}{2} P_\ell(\theta_k) P_\ell(\theta_j) = 0 , \tag{9}$$

if $k \neq j$.

So Eq. 6 gives

$$\phi_j = \frac{\sum_{\ell=0}^{n} \frac{2\ell+1}{2} f_\ell P_\ell(\theta_j)}{\sum_{\ell=0}^{n} \frac{2\ell+1}{2} \left[P_\ell(\theta_j)\right]^2} \quad . \tag{10}$$

Note that if $F(\mu)$ is normalized, then

$$\int_{-1}^{1} F(\mu)\, d\mu = 1 = \sum_{k=0}^{n} \phi_k \quad .$$

If all the ϕ_k's are positive, then a value of μ is selected from one of the θ_k's by choosing a random number, R, and letting $\mu = \theta_j$, if

$$\sum_{k=0}^{j-1} \phi_k < R \leq \sum_{k=0}^{j} \phi_k \quad . \tag{11}$$

If any of the ϕ_k's are negative, the selection technique is modified to select μ from

$$F'(\mu) = \sum_{k=0}^{n} \psi_k \, \delta(\theta_k - \mu) , \tag{12}$$

where

$$\psi_k = \frac{|\phi_k|}{\sum_{k=0}^{n} |\phi_k|} \tag{13}$$

and the neutron statistical weight is multiplied by the ratio

$$\left. \frac{F(\mu)}{F(\mu')} \right|_{\mu=\theta_k} = \frac{\phi_k}{\psi_k} \quad . \tag{14}$$

Note that this may yield negative weights, a situation which under some circumstances may give a negative value for an estimate. This indicates that an insufficient number of samples have been taken or that a higher order approximation is required.

APPENDIX
DERIVATION OF THE ORTHOGONALITY RELATION

Consider the term

$$I = (2\ell+1) P_\ell(\mu') P_\ell(\mu) (\mu' - \mu)$$
$$= (2\ell+1) \left[P_\ell(\mu)\, \mu' P_\ell(\mu') - P_\ell(\mu')\, \mu P_\ell(\mu)\right] . \tag{A1}$$

This expression can be rewritten using the recursion relationship

$$\mu' P_\ell(\mu') = \frac{1}{2\ell+1} \left\{(\ell+1) P_{\ell+1}(\mu') + \ell P_{\ell-1}(\mu')\right\} \tag{A2}$$

to get

$$I = P_\ell(\mu) \left[(\ell+1) P_{\ell+1}(\mu') + \ell P_{\ell-1}(\mu')\right] -$$
$$- P_\ell(\mu) \left[(\ell+1) P_{\ell+1}(\mu) + \ell P_{\ell-1}(\mu)\right]$$
$$= (\ell+1) \left[P_{\ell+1}(\mu') P_\ell(\mu) - P_\ell(\mu') P_{\ell+1}(\mu)\right] -$$
$$- \ell \left[P_\ell(\mu') P_{\ell-1}(\mu) - P_{\ell-1}(\mu') P_\ell(\mu)\right]$$
$$= j_\ell(\mu',\mu) - j_{\ell-1}(\mu',\mu), \tag{A3}$$

where

$$j_\ell(\mu',\mu) = (\ell+1) \left[P_{\ell+1}(\mu') P_\ell(\mu) - P_\ell(\mu') P_{\ell+1}(\mu)\right] . \tag{A4}$$

We sum both sides of Eq. A3 to get

$$(\mu' - \mu) \sum_{\ell=1}^{n} (2\ell+1) P_\ell(\mu') P_\ell(\mu) =$$
$$= \sum_{\ell=1}^{n} \left\{j_\ell(\mu',\mu) - j_{\ell-1}(\mu',\mu)\right\}$$
$$= \sum_{\ell=1}^{n} j_\ell(\mu',\mu) = \sum_{\ell=0}^{n-1} j_\ell(\mu',\mu)$$
$$= j_n(\mu',\mu) - (\mu' - \mu) . \tag{A5}$$

Hence

$$(\mu' - \mu) \sum_{\ell=0}^{n} (2\ell+1) P_\ell(\mu') P_\ell(\mu) = j_n(\mu',\mu) , \tag{A6}$$

or

$$\sum_{\ell=0}^{n} (2\ell + 1) P_\ell(\mu') P_\ell(\mu) = (n+1) \left[\frac{P_{n+1}(\mu') P_n(\mu) - P_n(\mu') P_{n+1}(\mu)}{\mu' - \mu} \right] , \qquad (A7)$$

which is Eq. 7.

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Some Tests of Coveyou's Anisotropic Selection Technique*

The application of R. R. Coveyou's anisotropic selection technique[1] and its proper inclusion into the Monte Carlo neutron-transport code[2] 05R have been tested by several calculations and comparisons. The technique is the result of its originator's initial attempt to devise a scheme for choosing from an anisotropic distribution, and so there was no older, tried technique against which either its accuracy or its effect on machine time could be tested for arbitrary angular distributions. Problems for which analytic solutions exist offer a better test of the accuracy of the technique than do problems solved by another Monte Carlo method which is itself subject to error in any case. For this reason one-velocity problems with isotropic scattering in the laboratory system were chosen for study with the anisotropy being introduced in the center-of-mass system in such a way as to result in isotropic laboratory scattering.

Beach *et al.*[3] have solved various one-velocity neutron-transport problems with isotropic scattering in the laboratory system by semianalytical methods. The problem chosen for comparison was the calculation of the flux from a plane isotropic source in a medium having a scattering cross section equal to half its total cross section. So that all parts of the code could be tested, the 05R calculations were made with constant cross sections, but with neutron slowing down permitted. The fluxes were obtained by a simple statistical estimation procedure which extended the path of a neutron from each collision point to the various planes at which the flux was desired, the contribution from each collision being given by

$$\frac{W}{|\mu|} e^{-|(x'-x)|/\lambda|\mu|} , \qquad (1)$$

where

- W = the statistical weight of the neutron after collision
- x' = the x coordinate of the plane at which the flux is to be estimated
- x = the x coordinate of the collision point
- λ = the total mean free path
- μ = the cosine of the angle between the neutron velocity vector and the x axis

and the source is in the y-z plane at the origin.

In order to isolate the systematic errors inherent in the Monte Carlo technique, the 05R calculations were made both for a medium whose scattering was isotropic in the center of mass and whose mass was 240, thus making the scattering in the laboratory system very nearly isotropic, and for a medium whose scattering was anisotropic in the center of mass. The latter medium was a half-and-half mixture of scatterers having masses of 2 and 3, each with a P_8 approximation to the center-of-mass scattering angular distribution which yields an isotropic-laboratory distribution. The only reason for using a mixture rather than a single scatterer was to test the code's ability to handle mixtures of anisotropic scatterers. Two thousand neutrons were run for each case.

The ℓ-th Legendre coefficient in an expansion of the center-of-mass angular distribution for a scatterer of mass A which gives an isotropic laboratory distribution is given by $(\ell+1)/(2\ell+1)(-1/A)^{\ell}$: The center-of-mass angular distribution resulting from a P_8 approximation for a mass 2 scatterer is given in Fig. 1, where the probability per unit cosine $F(\mu)$ is plotted as a function of the cosine of the center-of-mass scattering angle μ. Also shown as vertical bars are the ϕ_k's for the P_8 approximation required by Coveyou's method. They occur at the roots of P_9. It is somewhat startling to see the probability of selecting $\mu = 0.968$ fall below that of selecting $\mu = -0.836$ while $F(\mu)$ is rising. This may be qualitatively seen by interpreting the ϕ_k's as the integral of $F(\mu)$ over some appropriate interval including the k-th root. As the roots

*Research sponsored by the USAEC under contract with Union Carbide Corporation.

[1] R. R. COVEYOU, "A Monte Carlo Technique for Selecting Neutron Scattering Angles from Anisotropic Angular Distributions," *Nucl. Sci. Eng.* this issue, p. xxx.

[2] R. R. COVEYOU, J. G. SULLIVAN and H. P. CARTER, *Codes for Reactor Computations*, p. 267, International Atomic Energy Agency, Vienna, (1961).

[3] L. A. BEACH *et al.*, "Comparison of Solutions to the One-Velocity Neutron Diffusion Problem," NRL-5052 (Dec. 23, 1957).