$$\psi^+(a,\mu) - \frac{1}{\sigma}\mu\frac{\partial\psi^+(a,\mu)}{\partial z} - \frac{1}{\sigma}\sum_{\text{odd}}(2n+1)\frac{cf_n}{1-cf_n}P_n(\mu)\int_0^1 d\mu' P_n(\mu')\mu'\frac{\partial\psi^+(a,\mu')}{\partial z} = A_a(\mu) \tag{8a}$$

$$\psi^+(b,\mu) + \frac{1}{\sigma}\mu\frac{\partial\psi^+(b,\mu)}{\partial z} + \frac{1}{\sigma}\sum_{\text{odd}}(2n+1)\frac{cf_n}{1-cf_n}P_n(\mu)\int_0^1 d\mu' P_n(\mu')\mu'\frac{\partial\psi^+(b,\mu')}{\partial z} = A_b(\mu)\,. \tag{8b}$$

We denote

$$A_a(\mu) = A^+(\mu)$$
$$A_b(\mu) = A^-(-\mu)\,. \tag{9}$$

Now it is seen that Equation (6) with the boundary conditions $(8a,b)$ is completely equivalent to Equation (1) with the boundary conditions $(7a,b)$.

To get the total directional flux we use Equations (5) and (2). It is now easy to construct the functional which if set to be stationary would give Equation (6) as the Lagrange-Euler equation and Equations $(8a,b)$ as subsidiary conditions.

It will be

$$F[\psi^+] = \int_a^b dz \int_0^1 d\mu\left[-\mu^2\left(\frac{\partial\psi^+(z,\mu)}{\partial z}\right)^2 - \sigma^2(\psi^+(z,\mu))^2\right] + c\sigma^2\sum_{\text{even}}^{7}(2n+1)f_nP_n(\mu)\psi^+(z,\mu).$$

$$\int_0^1 d\mu' P_n(\mu')\psi^+(z,\mu')$$
$$-\sum_{\text{odd}}(2n+1)\frac{cf_n}{1-cf_n}\mu P_n(\mu)\frac{\partial\psi^+(z,\mu)}{\partial z}\int_0^1 d\mu' P_n(\mu')\mu'\frac{\partial\psi^+(z,\mu')}{\partial z}$$
$$-2\sigma\int_0^1 d\mu\,\mu\,\psi^+(b,\mu)\left[\frac{1}{2}\psi(b,\mu) - A_b(\mu)\right]$$
$$-2\sigma\int_0^1 d\mu\,\mu\psi^+(a,\mu)\left[\frac{1}{2}\psi^+(a,\mu) - A_a(\mu)\right]. \tag{10}$$

In the similar way one could have obtained the self-adjoint equation for $\psi^-(z,\mu)$ and the corresponding functional.

By way of illustration, let us take the trial function

$$\psi^+(z,\mu) = \frac{1}{2}\phi(z). \tag{11}$$

The functional given by Equation (10) takes the form:

$$F[\phi] = \frac{1}{4}\int_a^b dz\left[-\frac{1}{3(1-cf_1)}\left(\frac{d\phi(z)}{dz}\right)^2 - \sigma^2(1-c)(\phi(z))^2\right] - \frac{\sigma}{8}(\phi(b))^2 - \frac{\sigma}{8}(\phi(a))^2 + \sigma\phi(b)\bar{A}_b + \sigma\phi(a)\bar{A}_a \tag{12}$$

where

$$\bar{A}_a = \int_0^1 d\mu\ \mu\ A_a(\mu)$$
$$\bar{A}_b = \int_0^1 d\mu\ \mu\ A_b(\mu)\,. \tag{13}$$

Setting the functional given by Equation (12) stationary we get the diffusion equation

$$\frac{1}{3\sigma(1-cf_1)}\frac{d^2\phi(z)}{dz^2} - \sigma(1-c)\phi(z) = 0 \tag{14}$$

and the boundary conditions

$$J^+(a) = \frac{\phi(a)}{u} - \frac{1}{6\sigma}\frac{1}{1-cf_1}\frac{d\phi(a)}{dz} = \bar{A}_a$$
$$J^-(b) = \frac{\phi(b)}{u} + \frac{1}{6\sigma}\frac{1}{1-cf_1}\frac{d\phi(b)}{dz} = \bar{A}_b\,. \tag{15}$$

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Received October 24, 1963
Revised February 11, 1964

# Evaluation of Albedo of Neutrons for Slab Wall

## INTRODUCTION

Let us consider a neutron beam incident upon a wall. After collisions in the material of the wall, a fraction of the neutrons of the beam will be reflected. The effects of scattering in the wall will be estimated by consideration of multiple scattering. The ratio of the current of neutrons emerging from the surface of the wall to the incident current is called the albedo.

In this derivation we will assume that the scattering is isotropic and that there is no appreciable energy degradation of neutrons.

## SEMI-INFINITE WALL

Let us have a parallel beam of neutrons of strength $J_0$ incident upon the wall at right angles to its surface. It is assumed that the diameter of the beam is so large that the problem is only one dimensional. The thickness of the wall is assumed to be infinite.

Let $F(z)$ denote the neutron flux at a point in the wall located at distance $z$ from the surface of the wall. The function $F(z)$ satisfies the equation

$$F(z) = \frac{\Sigma_s}{2}\int_0^\infty F(z_1)\,E_1[\Sigma_t|z_1 - z|]dz_1 + J_0e^{-\Sigma_t z} \tag{1}$$

which is a well known form of the transport equation. Here $\Sigma_t$ is the macroscopic total cross section, $\Sigma_s$ denotes the macroscopic scattering cross section and the function $E_1[\Sigma_t|z_1-z|]$ is defined by

$$E_1[\Sigma_t|z_1-z|] = \int_{|z_1-z|}^{\infty} \frac{e^{-\Sigma_t u}}{u}\, du \, .$$

An exact solution of Eq. (1) seems quite difficult to obtain. However, a simple approximation yields a reasonable result. Let us express the equation (1) in the form

$$F(z) = \frac{\Sigma_s}{2}\int_0^z F(z_1)E_1[\Sigma_t(z-z_1)]dz_1 + \frac{\Sigma_s}{2}\int_z^{\infty} F(z_1)E_1[\Sigma_t(z_1-z)]dz_1 + J_0 e^{-\Sigma_t z} \qquad (2)$$

and consider the integral

$$\int_z^{\infty} F(z_1)E_1[\Sigma_t(z_1-z)]dz_1 \, . \qquad (3)$$

Let us represent the function $F(z)$ in the form of product

$$F(z) = F_0(z)\, e^{-k\Sigma_t z}$$

where the coefficient $k$ is to be found from the transcendental equation

$$\frac{\Sigma_s}{\Sigma_t}\,\frac{\tanh^{-1}k}{k} = 1 \, .$$

The form of the function $F(z)$ has been chosen with respect to the exact solution of transport equation in an infinite space $F(z) = C_1 e^{-k\Sigma_t z} + C_2 e^{k\Sigma_1 z}$

Consider the function $e^{-k\Sigma_t z_1} E_1[\Sigma_t(z_1-z)]$. At $z_1 = z$ the function is not bounded and drops off very rapidly as $(z_1 - z)$ increases. This sharp peaking makes the kernel act almost as a delta function. If the function $F_0(z_1)$ is assumed to be smoothly varying, the integral (3) may be approximated by

$$\int_z^{\infty} F(z_1)E_1[\Sigma_t(z_1-z)]dz_1 \approx F(z)\int_z^{\infty} e^{-k\Sigma_t(z_1-z)} E_1[\Sigma_t(z_1-z)]\, dz_1 = \frac{\ln(1+k)}{k\Sigma_t} F(z) \, . \qquad (4)$$

Introducing the relation (4) into the equation (2) we obtain

$$F(z) = \frac{\Sigma_s}{2}\int_0^z F(z_1)E_1[\Sigma_t(z-z_1)]\, dz_1 + \frac{\Sigma_s}{2\Sigma_t}\,\frac{\ln(1+k)}{k} F(z) + J_0 e^{-\Sigma_t z} \, . \qquad (5)$$

Let us integrate equation (5) over $z$ from 0 to $\infty$

$$\int_0^{\infty} F(z)dz = \frac{\Sigma_s}{2\Sigma_t}\int_0^{\infty} F(z)dz + \frac{\Sigma_s}{2\Sigma_t}\,\frac{\ln(1+k)}{k}\int_0^{\infty} F(z)dz + \frac{J_0}{\Sigma_t} \, . \qquad (6)$$

Contraction of equation (6) yields

$$\Sigma_t \int_0^{\infty} F(z)dz = \frac{J_0}{1 - \frac{\Sigma_s}{2\Sigma_t}\left(1 + \frac{\ln(1+k)}{k}\right)} \, . \qquad (7)$$

Let us now determine the current of neutrons $J$ emerging from the surface of the wall. Integrating equation (1) over $z$ from 0 to $\infty$ we obtain

$$\int_0^{\infty} F(z)dz = \frac{\Sigma_s}{2\Sigma_t}\int_0^{\infty} F(z_1)(2 - E_2[\Sigma_t z_1])dz_1 + \frac{J_0}{\Sigma_t}$$

Making use of the relation

$$J = \Sigma_t \frac{\Sigma_s}{2\Sigma_t}\int_0^{\infty} F(z_1)E_2[\Sigma_t z_1]dz_1 \, , \qquad (8)$$

we can write the equation (8) in the form

$$J = J_0 - \left(1 - \frac{\Sigma_s}{\Sigma_t}\right)\Sigma_t \int_0^{\infty} F(z)dz \, . \qquad (9)$$

Putting (7) into (9) we have the relation

$$J = J_0\left[1 - \frac{1 - \frac{\Sigma_s}{\Sigma_t}}{1 - \frac{\Sigma_s}{2\Sigma_t}\left(1 + \frac{\ln(1+k)}{k}\right)}\right] \, .$$

Hence the albedo is

$$A = \frac{J}{J_0} = 1 - \frac{1 - \frac{\Sigma_s}{\Sigma_t}}{1 - \frac{\Sigma_s}{2\Sigma_t}\left(1 + \frac{\ln(1+k)}{k}\right)} \, . \qquad (10)$$

## DISCUSSION OF ERRORS

Let us compare the numerical results computed from the formula (10) with more accurate results.

TABLE I

| $\frac{\Sigma_s}{\Sigma_t}$ | 1 - $A$ | |
|---|---|---|
| | "Exact" solution | Formula (10) |
| 0 | 1.000 | 1.000 |
| 0.25 | 0.955 | 0.951 |
| 0.35 | 0.930 | 0.924 |
| 0.45 | 0.902 | 0.890 |
| 0.55 | 0.865 | 0.848 |
| 0.65 | 0.820 | 0.793 |
| 0.75 | 0.752 | 0.720 |
| 0.85 | 0.660 | 0.611 |
| 0.95 | 0.464 | 0.409 |
| 0.98 | 0.327 | 0.280 |
| 0.99 | 0.247 | 0.207 |
| 1.00 | 0 | 0 |

Making use of the Wiener-Hopf technique[1] we obtain the exact solution of the albedo problem in the form

$$A = 1 - \frac{2\Sigma_t}{\Sigma_s}(1-k)\sqrt{1 - \frac{\Sigma_s}{\Sigma_t}}\ \phi(1) \qquad (11)$$

where the function $\phi(\mu)$ is to be found from the integral equation

$$\int_0^1 \frac{y\,\phi(y)}{y+\mu}\,dy = \frac{\frac{\Sigma_s}{\Sigma_t}}{2\phi(\mu)(1-k^2\mu^2)}\ . \qquad (12)$$

According to the approximation technique presented in Reference 1 we have computed the values of function $\phi(1)$ for some values of $\frac{\Sigma_s}{\Sigma_t}$. The error introduced by this approximation is less than 0.1%.

The comparison of the "albedo defects," 1-A, is given in Table I.

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Received January 5, 1962
Revised January 6, 1964

[1]YU. A. ROMANOV, "Exact Solutions of Single Velocity Kinetic Equation and Their Application in Calculating Diffusion Problems /Improved Diffusion Method/," FTD-TT-61-124.