P_1 Analytical Solution of the Energy- and Space-Dependent Transport Equation for Thermal Neutrons

INTRODUCTION

Starting from the fundamental theory given by Case[1] a number of investigations have been made in order to solve the transport equation with energy-dependent cross sections. The most general method seems to be that given by Bednarz and Mika[2] while the method given by Fuchs and Collatz[3] assumes an isotropic kernel. In both cases the final formulas are, however, complicated, and their applications are given for simple kernels.

As approximations in any case cannot be avoided we start in this paper with the P_1 approximations of the kernel and of the function to be found.

INFINITE MEDIUM

For thermal neutrons (the integration interval for fast neutrons is $[E, E/\alpha]$) in an infinite homogeneous medium of plane geometry the following equations hold:

$$\frac{\partial \Phi_1(z,E)}{\partial z} + \sigma(E)\Phi_0(z,E) = \int_0^\infty \sigma_0(E'\to E)\Phi_0(z,E')dE' + S_0(z,E) \tag{1}$$

$$\frac{1}{3}\frac{\partial \Phi_0(z,E)}{\partial z} + \sigma(E)\Phi_1(z,E) = \int_0^\infty \sigma_1(E'\to E)\Phi_1(z,E')dE' + S_1(z,E). \tag{1a}$$

The sources are assumed to be symmetrically distributed about the plane $z = 0$.

The boundary conditions satisfied by $\Phi_j(z,E)$ are

$$\Phi_j(\infty, E) = \Phi_j(z,\infty) = 0 \qquad (j = 0,1)$$

$$\frac{\partial \Phi_j(z,E)}{\partial z} = 0 \qquad \text{(symmetry)} .$$

After a z-Fourier transformation, separation[4] of $\Phi_0(z,E)$ and $\Phi_1(z,E)$, and some manipulation we obtain:

$$\begin{aligned}\Phi_0(k,E) = D(k,E)\Big[&\sigma(E)\int_0^\infty \sigma_0(E'\to E)\Phi_0(k,E')dE' + \\ &+ \int_0^\infty \sigma_1(E'\to E)\sigma(E')\Phi_0(k,E')dE' - \\ &- \int_0^\infty \sigma_1(E'\to E)\int_0^\infty \sigma_0(E''\to E')\Phi_0(k,E'')dE''dE' - \\ &- \int_0^\infty \sigma_1(E'\to E)S_0(k,E')dE' + \\ &+ \sigma(E)S_0(k,E) - \frac{1}{ik}S_1(k,E)\Big],\end{aligned} \tag{2}$$

where

$$D(k,E) = \frac{3}{3\sigma^2(E) + k^2} . \tag{3}$$

A similar form follows for $\Phi_1(k,E)$.

THE APPROXIMATION OF THE KERNELS

If $\{f_n(E);\ 0 \le E \le \infty;\ n = 1,2,\ldots\}$ is a complete system of orthonormal functions, then we can expand the kernels[a] $\sigma_j(E'\to E)$ as

$$\sigma_j(E'\to E) = \sum_{n=1}^{\infty} A_n^j(E)f_n(E') \qquad (j = 0,1), \tag{4}$$

where

$$A_n^j(E) = \int_0^\infty \sigma_j(E'\to E)f_n(E')dE' \qquad (n = 1,2,3,\ldots) .$$

For the existence of Eq. (4) it is necessary that the left-hand side of Eq. (4) possesses a second-order norm $N_{E'}\sigma_j(E'\to E)$ with respect to E', i.e.

$$\int_0^\infty |\sigma_j(E'\to E)|^2 dE' = \sum_{n=1}^{\infty} |A_n^j(E)|^2 \le N_{E'}\sigma_j(E'\to E)$$

(Bessel's inequality).

For the existence of $\sum_{n=1}^{\infty} |A_n^j(E)|^2$ it is sufficient that $N_{E'}\sigma_j(E'\to E)$ is bounded for every $E \ge 0$.

Now, integrals like

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[1] K. M. CASE, *Ann. Phys.*, **9**, 1-23 (1960).

[2] R. J. BEDNARZ and J. R. MIKA, *J. Math. Phys.*, **4**, 9, 1285-1292 (1963).

[3] K. FUCHS and S. COLLATZ, *Kernenergie* 6/7, 386-391 (1964).

[4] F. G. TRICOMI, *Integral Equations*, p. 150, Intersc. Publ. (1958).

[a] The kernels can be written as

$$\sigma_j(E'\to E) = \sigma_j(E')\frac{\sigma_j(E'\to E)}{\sigma_j(E')} = \sigma_j(E')\frac{\sum_{n=1}^{N} A_n^j(E)f_n(E')}{\sigma_j^N(E')},$$

where

$$\sigma_j^N(E') = \sum_{n=1}^{N}\left[\int_0^\infty A_n^j(E')dE'\right]f_n(E').$$

This normalization is necessary for the neutron conservation.

$$\left.\begin{array}{l}\int_0^\infty \sigma_j(E'\to E)^2 dE' , \\ \text{or} \quad \int_0^\infty dE \int_0^\infty \sigma_j(E'\to E)^2 dE'\end{array}\right\} \quad (5)$$

are in general divergent. Because of the distribution character of the function to be found, however, it is possible to regularize those integrals in the neighborhood of the singularities.

Consider the integral $\int_0^\infty \sigma_0(E'\to E)dE$.

This integral exists in any case and equals the total cross section σ_{scatt} of the medium. The integral $\int_0^\infty \sigma_{\mathrm{scatt}}(E')dE'$ naturally does not exist, and the divergence comes from an E' region where the functions to be found vanish approximately. In any case integrals like Eq. (5) may diverge and the divergences are of local character at $E,E' = 0$ and $E,E' = \infty$. These difficulties can be circumvented by regularizing the integrals.

A suitable regularization is

$$\sigma_j^*(E'\to E) = \frac{e^{-\rho_j E}\; e^{-\xi_j E'}\; E^{\rho_j^*}\; E^{\xi_j^*}}{(\rho_j^{**} + E^{\rho_j^*})(\xi_j^{**} + E'^{\xi_j^*})}\,\sigma_j(E'\to E) . \quad (6)$$

Here ρ_j^*, ρ_j^{**}, ξ_j^*, ξ_j^{**} are suitably chosen small positive constants and ρ_j and ξ_j are defined by

$$\frac{1}{\xi_j},\ \frac{1}{\rho_j} \geq E_{0j} ,$$

where E_{0j} is an energy beyond which the functions Φ_j vanish approximately. In this way it is possible to see that expansions (4) are meaningful.

EXPLICIT FORM OF THE APPROXIMATE SOLUTION

It follows from Eqs. (2) and (4), if one retains only N terms in the expansion, that

$$\begin{aligned}\Phi_j(k,E) = D(k,E) \sum_{n=1}^{N} \Big\{ &\sigma(E)A_n^j(E)P_n^j(k) + \\ &+ [\delta_{0j}A_n^1(E) + \delta_{1j}A_n^0(E)][Q_n^j(k) - S_n^j(k)] - \\ &- \sum_{m=1}^{N} [\delta_{0j}A_n^1(E) + \delta_{1j}A_n^0(E)]R_{nm}^j\, P_n^j(k)\Big\} \times \\ &\times \Big\{- ik\Big[\delta_{0j}S_1(k,E) + \tfrac{1}{3}\delta_{1j}S_0(k,E)\Big] + \\ &+ \sigma(E)S_j(k,E)\Big\} D(k,E). \end{aligned} \quad (7)$$

In Eq. (7) we have

$$P_n^j(k) = \int_0^\infty \Phi_j(k,E)f_n(E)dE \quad (8)$$

$$Q_n^j(k) = \int_0^\infty \Phi_j(k,E)\sigma(E)f_n(E)dE \quad (9)$$

$$S_n^j(k) = \int_0^\infty S_j(k,E)f_n(E)dE \quad (10)$$

$$R_{nm}^j = \int_0^\infty A_m^j(E)f_n(E)dE \quad (11)$$

δ_{0j}, δ_{1j} = Kronecker-Delta symbols.

The terms $S_n^j(k)$ and R_{nm}^j are known quantities while $P_n^j(k)$ and $Q_n^j(k)$, which determine our solutions, $\Phi_j(k,E)$, have to be found. Inserting Eq. (7) into Eqs. (8) and (9) one obtains in writing the result in matrix form (I is the unit matrix)

$$\left(\begin{array}{c|c} {}^0T^j - \mathrm{I} & {}^0M^j \\ \hline {}^1T^j & {}^1M^j - I \end{array}\right)\begin{pmatrix} P_1^j \\ P_2^j \\ \vdots \\ \vdots \\ P_{2N}^j \end{pmatrix} = \begin{pmatrix} {}^0H_1^j \\ {}^0H_2^j \\ \vdots \\ {}^0H_N^j \\ {}^1H_1^j \\ \vdots \\ {}^1H_N^j \end{pmatrix}, \quad (12)$$

where

$$P_{N+n}^j(k) = Q_n^j(k) \quad (n = 1,2, \ldots . N) \quad (13)$$

$$\begin{aligned}{}^\ell T_{nn'}^j(k) = {}^{\ell+1}L_{nn'}^j(k) - \\ - \sum_{m=1}^{N}\Big[\delta_{0j}{}^\ell L_{mn'}^1(k) + \delta_{1j}{}^\ell L_{mn'}^0(k)\Big]R_{nm}^j\end{aligned} \quad (14)$$

$${}^\ell M_{nn'}^j(k) = \delta_{0j}{}^\ell L_{nn'}^1(k) + \delta_{1j}{}^\ell L_{nn'}^0(k) \quad (15)$$

$$\begin{aligned}{}^\ell H_{n'}^j(k) = \delta_{0j}\Big[\sum_{n=1}^{N} S_n^j(k)\,{}^\ell L_{nn'}^1(k) - ik\,{}^\ell X_{n'}^1(k)\Big] + \\ + \delta_{1j}\Big[\sum_{n=1}^{N} S_n^j(k)\,{}^\ell L_{nn'}^0(k) - ik\,{}^\ell X_{n'}^0(k)\Big] - \\ - {}^{\ell+1}X_n(k) \quad (\ell = 0,1;\ j = 0,1)\end{aligned} \quad (16)$$

$$^{\ell}L^{j}_{nn'}(k) = \int_0^\infty \sigma^{\ell}(E)D(k,E)A^{j}_{n}(k)f_{n'}(E)dE \qquad (17)$$

$$^{\ell}X^{j}_{n'}(k) = \int_0^\infty \sigma^{\ell}(E)D(k,E)S_j(k,E)f_{n'}(E)dE. \qquad (18)$$

It is useful to choose the set of functions $\{f_n(E)\}$ in such a way that a rapid convergence of the expansions (4) can be achieved, given that all expressions (14) to (18) can always be found as explicit functions of k for a certain N.

Using Cramer's rule we obtain from Eq. (12)

$$P^{j}_{n}(k) = \frac{\sum_{n'=1}^{2N} |(W^{j})^{nn'}| H^{j}_{n'}}{|W^{j}|},$$

$$\begin{pmatrix} H^{j}_{N+n} = {}^{1}H^{j}_{n};\ H^{j}_{n} = {}^{0}H^{j}_{n} \\ n = 1,2,\ldots N \end{pmatrix} . \qquad (19)$$

From Eqs. (7) and (19), it follows that

$$\Phi_j(z,E) = \sum_{n=1}^{N}\left\{\left[A^{j}_{n}(E) - V^{jj'}_{n}(E)\right]B^{j}_{n}(z,E) + \right.$$

$$\left. + A^{j}_{n}(E)B^{j}_{N+n}(z,E)\right\} + K_j(z,E), \qquad (20)$$

where

$$V^{jj'}_{n}(E) = \sum_{m=1}^{N} R^{j}_{nm} A^{j'}_{n}(E);\ j \neq j' = 0,\ 1 = j, \qquad (21)$$

$$B^{j}_{n}(z,E)$$

$$= \sqrt{\frac{3\pi}{2}}\, |W^{j}_{\sigma}|^{-1} \sum_{n'=1}^{2N} |(W^{j}_{\sigma})^{nn'}| H^{j}_{n'} \frac{\exp[-\sqrt{3}\sigma(E)|z|]}{\sigma(E)} +$$

$$+ 3i\sqrt{\pi}\sum_{v=1}^{2}\sum_{\alpha=1}^{2N} |{}^{v}W^{j}_{k\alpha}|^{-1} \sum_{n'=1}^{2N} |({}^{v}W^{j}_{k\alpha})^{nn'}| H^{j}_{n'} \times$$

$$\times \frac{\exp[-|{}^{v}k''_{\alpha}||z|]}{{}^{v}k^{2}_{\alpha} + 3\sigma^{2}(E)} \qquad (22)$$

$$(n = 1,2,3,\ldots .\ 2N)$$

and

$$K_j(z,E) = \frac{3}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sigma(E)\frac{S_0(k,E)}{3\sigma^2(E)+k^2} - \frac{ik}{2j+1}S_1(k,E) - \right.$$

$$\left. - Y^{jj'}(k,E)\right] e^{ikz}\, dk \qquad (23)$$

with

$$Y^{jj'}(k,E) = \sum_{n=1}^{N} S^{j}_{n}(k)A^{j'}_{n}(E);\ \ j \neq j' = 0,\ 1 = j\ .$$

In Eq. (22) ${}^{v}k_{\alpha}$ $(\alpha = 1,2,\ldots 2N)$ are the zeros of $|W^{j}(k)|$ with

$${}^{v}k'_{\alpha} = \mathrm{Re}({}^{v}k_{\alpha})$$

and

$${}^{v}k''_{\alpha} = \mathrm{Im}({}^{v}k_{\alpha})\ .$$

Assuming that the source has the form

$$S_j(z,E) = S_j(E)\delta(z) \quad (j = 0,1)\ ,$$

Eq. (23) takes the form

$$K_j(z,E) = \frac{3}{\sqrt{2\pi}}\left[S_j(E) + \frac{3}{2j+1}S_{j'}(E) - \right.$$

$$\left. - \frac{1}{\sigma(E)}\sum_{n=1}^{N} S^{j}_{n}A^{j}_{n}(E)\right] e^{-\sqrt{3}\sigma(E)|z|}\ .$$

CONCLUSIONS

We have considered the solution of the transport equation under the following conditions:

a) homogeneous infinite medium of plane geometry

b) sources distributed symmetrically about the $(z = 0)$ plane

c) P_1 approximation of the scattering kernel

d) P_1 approximation of the solution.

A special feature of the method given here is that the kernels $\sigma_0(E'\rightarrow E)$ and $\sigma_1(E'\rightarrow E)$ can be expanded as in formula (4). This presupposes the possibility of regularizing the integrals (5) so that a norm exists. Beyond this possibility one can always approximate the kernels by polynomials according to the Weierstrass theorem. The polynomials are degenerate.

It is observed that this method is convenient for practical calculations. The only quantities which have to be calculated numerically are simple integrals involving products of known functions and cross sections. These integrals constitute in a certain manner the elements of a determinant, the zeros of which are the poles of some integrands.

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