Letter to the Editor

Transport Calculations in Cylindrical Geometry

In past years, the solution of the one-speed integral transport equation in homogeneous one-dimensional bodies has attracted many researchers, and several methods have been developed for dealing with it, among them the integral transform method[1] (IT_N) and the Carlvik spatial-spherical harmonics (SSH) method.[2]

Recently, a new analytical method developed by Milgram[3] has been applied to the solution of the critical and surface-source problems in cylindrical geometry with isotropic scattering. Questions regarding differences between this analytical method (henceforth referred to as A_N) and the IT_N have been raised[3] but not fully resolved. Within the framework of the general approximation theory, both methods are essentially equivalent, as is shown here. Numerical comparisons of the two approaches also show that the IT_N method has a better rate of convergence.

To prove the equivalence, we first review the Galerkin-Petrov method.[4] We consider an equation of second kind:

$$\lambda\Phi = H\Phi + S \quad , \tag{1}$$

where H is an operator acting in a separable Hilbert space X, and we let $B = \{\phi_i, i = 1, 2, \ldots\}$ and $B^* = \{\phi_i^*, i = 1, 2, \ldots\}$ be two complete linearly independent sets. Then, denoting by X_N and X_N^* the linear spans of the N first elements of B and B^*, respectively, the Galerkin-Petrov approximation of Eq. (1) in subspace X_N is

$$P_N(\lambda_N \Phi^N - H\Phi^N - S) = 0 \quad , \tag{2}$$

where $\Phi^N \in X_N$ and $P_N: X \to X_N^*$ is the orthogonal projection associated with subspace X_N^*. If $S \neq 0$, then $\lambda_N = \lambda$.

In the following, we confine ourselves to the special case in which B^* is taken to be the dual set of B, i.e., $(\phi_i^*, \phi_j) = \delta_{ij}$, where $(,)$ is the scalar product in X. Equation (2) can also be written as

$$\lambda_N \Phi_i^N = \sum_{j=1}^{N} (P_{ij} \Phi_j^N + S_i) \quad , \qquad (i = 1, \ldots, N) \quad . \tag{3}$$

Here, Φ_i^N and S_i are the components of Φ^N and S on the basis $\{\phi_i, i \leq N\}$ of subspace X_N, and the matrix elements are given by

$$P_{ij} = (\phi_i^*, H\phi_j) \quad . \tag{4}$$

Now let us consider the above technique as applied to the steady-state one-speed integral transport equation with isotropic sources and scattering in a homogeneous body. We take X as the set of square integrable functions for the scalar product

$$(f,g) = \int_V f(\boldsymbol{r}) g(\boldsymbol{r}) d\boldsymbol{r} \quad ,$$

where V is the volume of the body. Thus, henceforth, Eq. (1) is regarded as the integral equation with operator

$$H\Phi = \int_V \frac{\exp(-\Sigma|\boldsymbol{r} - \boldsymbol{r}'|)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|^2} \Phi(\boldsymbol{r}') d\boldsymbol{r}' \quad , \tag{5}$$

where Σ is the total macroscopic cross section. Note that the compactness of H assures the convergence of the Galerkin-Petrov approximation.[4]

Milgram's A_N method is readily cast into the form of a Galerkin-Petrov approximation. Indeed, the method is based on the expansion of S, Φ, and $H\Phi$ into series of even powers of r (distance to the axis of the cylinder), the approximation of order N being obtained by equating the coefficients of r^{2n} for $n < N$. Therefore, the A_N method can be viewed as a Galerkin-Petrov approximation of Eq. (1) with the nonorthogonal bases $B = \{\phi_i = r^{2(i-1)}, i = 1, 2, \ldots\}$.

On the other hand, even though the IT_N method solves the Fourier transform of Eq. (1) by expanding the kernel of the transformed integral operator into a complete set of orthonormal functions $\{\chi_i, i = 1, 2, \ldots\}$, its equations are step-by-step Fourier images of Eqs. (2), (3), and (4). The method has been shown to be completely equivalent to Carlvik's SSH method, which uses an orthonormal set, $B = B^*$, to construct a Galerkin-Petrov approximation of the integral transport equation[5]; the equivalence is based on the fact that the IT_N expansion functions are Fourier transforms of those of the SSH method: $\chi_i = \widehat{\phi_i \theta}$, where the caret indicates the Fourier transform and θ is the characteristic function of volume V. Legendre polynomials are currently utilized as the base functions for calculations in homogeneous slabs and spheres, and, for a cylinder of radius a, the orthonormal basis B is the set of Legendre polynomials of argument[2,6]

$$\left(1 - 2\frac{r^2}{a^2}\right) \quad .$$

[1] I. CARLVIK, *Nucl. Sci. Eng.*, **31**, 295 (1968).
[2] H. HEMBD, *Nucl. Sci. Eng.*, **40**, 224 (1970).
[3] M. S. MILGRAM, *Nucl. Sci. Eng.*, **68**, 249 (1978).
[4] M. A. KRASNOSEL'SKII et al., *Approximate Solution of Operator Equations*, Wolters-Noordhoff Publishing, Groningen (1972).
[5] J. MIKA and R. STANKIEWICZ, *Nucl. Sci. Eng.*, **36**, 450 (1969).
[6] R. SANCHEZ, "Généralisation de la méthode d'Asaoka pour le traitement d'une loi de choc linéairement anisotrope: données de référence en geometrie cylindrique," CEA-N-1831, Centre d'Etudes Nucléaires de Saclay (1975).

Consequently, the IT_N and A_N methods solve the integral transport equation in a homogeneous cylinder by using the same projective technique, but with different bases: While the former uses an orthonormal basis, the latter does not. In both methods, the subspace of approximation X_N is the space of polynomials of r^2 of degree smaller than N, and the approximate solution ΦN is obtained by making the residual of Eq. (1),

$$R(\Phi^N) = \lambda_N \Phi^N - H\Phi^N - S \quad ,$$

orthogonal to the "dual" space X_N^*. For the IT_N method, $X_N^* = X_N$ and the residual must be a function orthogonal to all polynomials of r^2 of degree smaller than N, whereas for the A_N method, $X_N^* \neq X_N$, and the residual has to be a linear combination of powers of r^2 of degree greater than N.

The preceding remarks also show that if, as suggested in Ref. 3, Legendre polynomials are implemented in the A_N method, the results so obtained will be identical to those that would be obtained with the IT_N method.

A comparison between the two methods can be established at two levels:

1. the relative rate of convergence with N of the approximate solutions to the exact one
2. the possibility of using symmetry properties of the transport operator to reduce the amount of numerical work involved in the calculations of the collision matrix elements P_{ij}.

A comparison of convergence rates is shown in Tables I and II, in which IT_N results for the critical and the surface-source problems are compared to their A_N counterparts. The IT_N calculations have been performed with a previously developed program,[6] and the A_N values have been taken from Ref. 3. Comparison of the critical eigenvalue λ and the rod average flux shows that, for the set of calculations considered, the IT_N method converges faster than the A_N.

With isotropic scattering, the integral transport operator is symmetric:

$$(f, Hg) = (g, Hf) \quad .$$

Thus, a numerical advantage can be gained when an orthonormal basis is used, as with the IT_N method. This results in the matrix reciprocity relations,

$$P_{ij} = P_{ji} \quad , \tag{6}$$

TABLE I

Values of the Critical Eigenvalue λ as a Function of the Optical Radius τ

τ	A_4	A_9	IT_4	IT_9
0.01	0.013299	0.013280	0.01327337	0.01327339
0.10	0.11778	0.11760	0.1175457	0.1175458
0.30	0.29228	0.29184	0.2916970	0.2916974
0.66862	0.50098	0.50025	0.5000022	0.5000031
0.80743	0.55663	0.55583	0.5555558	0.5555568
1.02085	0.62619	0.62531	0.6250019	0.6250031
2.0	0.80566	0.80468	0.8043225	0.8043243
3.57744	0.91027	0.90942	0.9090907	0.9090927
5.41152	0.95330	0.95265	0.9523826	0.9523843
9.04458	0.98093	0.98059	0.9803962	0.9803973

TABLE II

Values of the Average Flux for the Benchmark Black Rod Problem

(τ = optical radius, c = number of secondaries per collision)

τ	c	A_8	A_{34}	IT_8	IT_{34}
0.5	0.0	0.75775	0.75873	0.758791214	[a]
	0.2	0.82365	0.82452	0.824582113	0.824582108
	0.4	0.90257	0.90331	0.903361377	0.903361370
	0.6	0.99885	0.99941	0.999451305	0.999451297
	0.8	1.1190	1.1193	1.11934078	1.11934077
	1.0	1.2732	1.2732	1.27323954	[a]
5.0	0.0	0.11594	0.12568	0.1263404	[a]
	0.2	0.13931	0.14894	0.1495917	0.1495913
	0.4	0.17471	0.18410	0.1847411	0.1847402
	0.6	0.23578	0.24475	0.2453634	0.2453622
	0.8	0.37384	0.38192	0.3824921	0.3824908
	1.0	1.2732	1.2732	1.273240	[a]

[a] The IT_N values for $c = 0.0$ and $c = 1.0$ are exact at any order N.

which nearly cut in half the number of these computations. This is not the case for the A_N method, for which it is not even possible to write a matrix element as a finite linear combination of other matrix elements, due to the fact that ϕ_i^* is not a polynomial of finite degree.

We wish to further compare the IT_N and A_N methods for the surface-source problem with incoming isotropic angular flux ν, for which it is possible to obtain the corresponding source terms, $S_i = q_i$, from the matrix elements. This is done by using the conservation equation for uncollided neutrons,[7]

$$\int_{\partial V} \boldsymbol{J} \cdot d\boldsymbol{S} = \int_V (S - \Sigma\Phi) d\boldsymbol{r} \quad . \tag{7}$$

(Here, $\boldsymbol{J}$ and Φ are, respectively, the uncollided current and scalar flux produced by source S.) With $S = \phi_i^*$, we obtain

$$q_i = \frac{4\pi\nu}{a_1}(\delta_{i1} - \Sigma P_{i1}) \quad , \tag{8}$$

where we have assumed that the first representation function is constant:

$$\phi_1 = a_1 \quad .$$

This result is valid for the two methods. In particular, for the A_N method, $a_1 = 1$, and, with the source normalization of Ref. 3,

$$\nu = \frac{1}{2\pi^2 a} \quad .$$

The resulting relation, Eq. (8), can be verified using Eqs. (20) and (23) of Ref. 3.

Still another comparison arises from the possibility of reducing the problem with linearly anisotropic scattering,

$$\Sigma_s(\mu) = \frac{1}{4\pi}(\Sigma_s + \mu\Sigma_s^1) \quad ,$$

to the isotropic scattering problem. As has been shown elsewhere,[6] the Galerkin-Petrov approximation for the linearly

[7] R. SANCHEZ, *Nucl. Sci. Eng.*, **64**, 384 (1977).

anisotropic scattering problem is given by the system of equations,

$$\lambda_N \Phi_i^N = \sum_{j=1}^{N} (P_{ij} F_j^N + \tilde{P}_{ij} B_j^N) + q_i \quad , \quad (i = 1, \ldots, N) \quad , \tag{9a}$$

written in notation slightly different from that of Eq. (1). In this equation,

$$F_i^N = \Sigma_s \Phi_i^N + S_i \quad ,$$
$$B_i^N = \Sigma_s^1 [S_i - (\Sigma - \Sigma_s)\Phi_i^N] + S_i^1 \quad ,$$

and S_i and S_i^1 are, respectively, the components of the isotropic and anisotropic volume sources. The anisotropic matrix elements $\tilde{P}_{ij}$ can be written in terms of the isotropic matrix elements[6]:

$$\tilde{P}_{ij} = \sum_k \lambda_j^k P_{ik} - \lambda_j^i \quad , \tag{9b}$$

where, for the cylinder,

$$\lambda_j^k = \left[\phi_k^*, \int_r^a \frac{dr'}{r'} \int_0^{r'} \phi_j(r'') r'' dr'' \right] \quad . \tag{9c}$$

For the IT_N method, these coefficients vanish except for $k = j - 1$, j, or $j + 1$ (Ref. 6). Similarly, for the A_N method, one finds

$$\lambda_j^k = \frac{1}{(2j)^2} (a^{2j} \delta_{k1} - \delta_{k,j+1}) \quad . \tag{9d}$$

So, for both methods, it is possible to solve the linearly anisotropic scattering problem at order N with the isotropic matrix of order $N + 1$.

Although no definitive proof has been given as to which method has the better rate of convergence, the IT_N method gives the best results in all the calculations that we have made. Both methods share the numerical advantages shown in Eqs. (8) and (9), but only the IT_N method has the important matrix symmetry of Eq. (6).

We think that these facts are sufficient evidence to support the general conclusion that Galerkin-Petrov approximations of the integral transport equation based on orthonormal functions are superior to those utilizing nonorthogonal expansions.

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