$$\rho_{m\lambda k} = C_{mk} \sqrt{\frac{2\lambda + 1}{4\pi} \frac{(\lambda - m)!}{(\lambda + m)!}} P_{\lambda m}(\Gamma_{mk}). \quad (9)$$

The C_{mk} are determined by the normalization of the columns of ρ_m to unit length. Thus

$$C_{mk}^{-2} = \sum_{\lambda=0}^{L} \frac{2\lambda + 1}{4\pi} \frac{(\lambda - m)!}{(\lambda + m)!} P_{\lambda m}^2(\Gamma_{mk}). \quad (10)$$

The eigenvalues, Γ_{mk}, are determined by the requirement that, in an L-th order truncation,

$$\rho_{m,L+1,k} = 0 \qquad \text{or} \qquad P_{L+1,m}(\Gamma_{mk}) = 0.$$

The series for C_{ok}^{-2} can be summed by noting that[4]

$$\sum_{\lambda=0}^{L} (2\lambda + 1) P_\lambda(\omega_1) P_\lambda(\omega_2) \quad (11)$$

$$= \frac{L + 1}{\omega_1 - \omega_2} [P_{L+1}(\omega_1) P_L(\omega_2) - P_{L+1}(\omega_2) P_L(\omega_1)].$$

Taking limits as $\omega_2 \to \omega_1 = \Gamma_{ok}$ in Equation (11), and using Equation (10), we find

$$\sum_{\lambda=0}^{L} (2\lambda + 1) P_\lambda^2(\Gamma_{ok}) = (L + 1) P'_{L+1}(\Gamma_{ok}) P_L(\Gamma_{ok}). \quad (12)$$

But

$$P'_{L+1}(\Gamma_{ok}) = (1 - \Gamma_{ok}^2)^{-1} (L+1) [P_L(\Gamma_{ok}) - \Gamma_{ok} P_{L+1}(\Gamma_{ok})]. \quad (13)$$

Substituting Equation (13) in Equation (12) gives

$$\sum_{\lambda=0}^{L} (2\lambda + 1) P_\lambda^2(\Gamma_{ok}) = (1 - \Gamma_{ok}^2)^{-1} (L+1)^2 P_L^2(\Gamma_{ok}) = 4\pi C_{ok}^{-2}. \quad (14)$$

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Received March 6, 1963
Revised August 22, 1963

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A Note on the Measurement of the Transport Mean Free Path of Thermal Neutrons in Graphite by a Poison Method*

J. M. Hendrie *et al.*,[1] have reported a measurement of the transport mean free path of thermal neutrons, λ_t, in graphite by a copper-poison technique which is analogous to the heavy-water/boron technique described earlier by S. W. Kash and D. C. Woods.[2] The value of $\lambda_t = 2.77 \pm 0.05$ cm reported by Hendrie is somewhat larger than the value of 2.65 ± 0.03 cm inferred from measurements by E. Starr and G. Price[3] for the same AA Graphite, using pulsed-neutron techniques. In addition, other measurements by the pulsed-neutron method (all referred to a graphite density of 1.60 g/cm^3) are tabulated in Table I and are seen to have an average value of approximately 2.59 cm. Measurements of λ_t by extrapolation distance, complex diffusion length, and averaged cross sections are also indicated in Table I, but are not considered to be as reliable as either the pulsed-neutron method or the poison method.

Recent measurements of the diffusion-cooling constant, c, as shown in Table II, indicate that the value is probably in the neighborhood of 38×10^5 cm^4/sec, rather than that of 12 or 16×10^5 cm^4/sec, which were reported earlier.[4,5] The transport mean free path reported by Hendrie *et al.*,[1] was not corrected for the effect of diffusion cooling because that effect was considered to be negligible. However, in view of the large values of c recently reported, the value of λ_t as measured by the copper-poison method has been recomputed and is reported herein.

*Work done under the auspices of the U.S.A.E.C.

[1] J. M. HENDRIE, J. P. PHELPS, G. A. PRICE, E. V. WEINSTOCK, "Slowing Down and Diffusion Lengths of Neutrons in Graphite-Bismuth Systems", *Proc. of 2nd United Nations Int. Conf. on the Peaceful Uses of Atomic Energy, Geneva*, P/601 (1958).

[2] S. W. KASH and D. C. WOODS, "Measurements of the Transport Mean Free Path of Thermal Neutrons in D_2O by a Boron Poisoning Method", *Phys. Rev.* 90, 564 (1953).

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[4] A. V. ANTONOV *et al.*, "A Study of Neutron Diffusion in Beryllium, Graphite, and Water by Impulse Method", *Proc. of the Int. Conf. on the Peaceful Uses of Atomic Energy, Geneva*, P/661 (1956).

[5] K. H. BECKURTS, "Measurements with a Pulsed Neutron Source", *Nuc. Sci. & Eng.*, 2, 516 (1957).

TABLE I.

Transport Mean Free Path of Thermal Neutrons in Graphite[a]

$\frac{\lambda_t}{\text{cm}}$	Method	Reference
2.66 ± 0.23	extrapolation distance	A. V. Antonov[4]
2.74 ± 0.03 (AA)	extrapolation distance	J. M. Hendrie[1]
2.74 ± 0.11	averaged cross section	D. J. Hughes[6]
2.55 ± 0.09	complex diffusion length	V. Raievski[7]
2.49 ± 0.04	pulsed neutron	A. V. Antonov[4]
2.58 ± 0.02 (I)	pulsed neutron	K. H. Beckurts[5]
2.56 ± 0.03 (II)	pulsed neutron	K. H. Beckurts[5]
2.62 ± 0.02 (III)	pulsed neutron	K. H. Beckurts[5]
2.65 ± 0.04	pulsed neutron	M. Sagot[8]
2.58 ± 0.02	pulsed neutron	H. Klose[9]
2.59 ± 0.01 (GBF)	pulsed neutron	E. Starr[3]
2.65 ± 0.03 (AA)	pulsed neutron	E. Starr[3]
2.77 ± 0.05[b] (AA)	poison method	J. M. Hendrie[1]
2.62 ± 0.04[c] (AA)	poison method	J. M. Hendrie[1]

[a]All values corrected to graphit density = 1.60 g/cm^3.
[b]Not corrected for diffusion cooling.
[c]Corrected for diffusion cooling.

TABLE II.

Diffusion-Cooling Constant of Graphite

$C \times 10^5$ cm^4/sec	Method	Reference
12.0 ± 1.9	pulsed neutron	A. V. Antonov[4]
16.3 ± 2.5	pulsed neutron	K. H. Beckurts[5]
13.4 ± 3.3	Ag transmission	K. H. Beckurts[5]
26 ± 5	pulsed neutron	H. Klose[9]
37.9 ± 4	pulsed neutron	M. Sagot[8]
38 ± 5	pulsed neutron ($\bar{v}$)	E. Starr[10]
34 ± 3 (GBF)	pulsed neutron	E. Starr[3]
41 ± 4 (AA)	pulsed neutron	E. Starr[3]

The effect of diffusion cooling in the poison method can be seen by examining the neutron decay probability, λ_i, which describes the decay rate of neutrons in an absorbing and diffusing medium with macroscopic absorption cross section, Σ_a, mean free path, λ_t, and buckling, B_i^2,

$$\lambda_i = \overline{\Sigma_a v} + \frac{\overline{\lambda_t v}}{3} B_i^2 - cB_i^4 \quad . \tag{1}$$

Since the copper-poison method involves a steady state condition with an external source of neutrons, $\lambda_i = 0$, and $B_i^2 = -1/L^2$, where L is the diffusion length of thermal neutrons. Thus, equation (1) can be transformed to the following form for the exponential experiment:

$$0 = \overline{\Sigma_a v} - \frac{\overline{\lambda_t v}}{3} (1/L^2) - c(1/L^2)^2 . \tag{2}$$

If one assumes that $\Sigma_a \propto 1/v$ and that λ_t = constant, the averages over the products can be removed and replaced by an average over the neutron velocity $\bar{v}$, Σ_a now refers to the macroscopic cross section at the mean neutron velocity, $\bar{v}$. Differentiation of equation (2) with respect to Σ_a yields the following solution for λ_t:

$$\lambda_t = 3\left\{\left[\frac{\partial(1/L^2)}{\partial \Sigma_a}\right]^{-1} - \frac{2c}{\bar{v}} (1/L^2)\right\} \tag{3}$$

Equation (3), consequently, is the basis for the measurement of λ_t by the poison method, in which the variation of diffusion length, L, is observed as a function of poison cross section, Σ_a. In the case of $c = 0$, equation (3) reduces to:

$$\lambda_t = 3\left[\frac{\partial(1/L^2)}{\partial \Sigma_a}\right]^{-1} . \tag{4}$$

The data of Hendrie *et al.*,[1] have been recomputed with a diffusion-cooling constant equal

[6]D. J. HUGHES and R. B. SCHWARTZ, "Neutron Cross Sections", BNL-325 (1958).

[7]V. RAIEVSKI and J. HOROWITZ, "Determination of the Mean Transfer Free Path of Thermal Neutrons by Measurement of the Complex Diffusion Length", *Proc. of the Int. Conf. on the Peaceful Uses of Atomic Energy, Geneva,* P/360 (1956).

[8]M. SAGOT and H. TELLIER, private communication (1963).

[9]H. KLOSE, M. KUCHLE, and W. REICHARDT, "Pulsed Neutron Measurements on Graphite", *Proc. of the Brookhaven Conf. on Neutron Thermalization,* Vol. 3, 935, BNL-719 (C-32) (1962).

[10]E. STARR and J. W. DeVILLIERS, "Determination of Diffusion Cooling in Graphite by Measurement of the Average Neutron Velocity", *Proc. of the Brookhaven Conf. on Neutron Thermalization,* Vol. 3, 997, BNL-719 (C-32) (1962).

to 38×10^5 cm⁴/sec. They yield a corrected value of λ_t equal to 2.62 ± 0.04 cm at a graphite density of 1.60 g/gm³, which can be compared with the value of 2.65 ± 0.03 cm obtained by pulsed-neutron methods with the same graphite. The effect of this correction upon the previously reported graphite absorption cross section of 3.44 ± 0.08 mb[1] is negligible since the correction is a minimum at long relaxation lengths.

It appears that the transport mean free path of thermal neutrons in graphite as measured by the poison method is in agreement with the value obtained by the pulsed-neutron method, after due account is made for diffusion-cooling effects.

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Received September 13, 1963

Degenerate Solutions to the Transport Equation With Anisotropic Scattering

Following the work of Case,[1] Mika[2] has shown that a complete set of eigenfunctions to the one-velocity transport equation for plane symmetry can be found when the scattering function may be expanded in a finite series of Legendre polynomials.

It may happen that some of the solutions to the homogeneous transport equation, corresponding to multiple order eigenvalues, are degenerate. In order to find the additional solutions necessary to complete the set of eigenfunctions, one has to take derivatives of the degenerate eigenfunction with respect to the eigenvalue (cf. Eq. B.5, p. 425, of Reference 2). In particular, when the eigenvalue ν_0 is a root of order two, one solution is $\psi_{\nu_0}(x,\mu)$, while according to Equation B.5, a second solution is

$$\psi_1(x,\mu) = \frac{\partial}{\partial\left(\frac{1}{\nu}\right)}\left[e^{-x/\nu}\ \phi_\nu(\mu)\right]_{\nu=\nu_0}. \qquad (1)$$

It is the purpose of this note to point out that this procedure is not always valid for anisotropic scattering and to indicate where the difficulty arises for a particular case. For a nonabsorbing medium ($c = 1$) there is always a double root at infinity, and the second solution corresponding to this root, as given by Equation (1), is correct for isotropic scattering, but incorrect for anisotropic scattering.

For simplicity, consider the case of linear anisotropic scattering. Let L be the linear transport operator for this case.

$$L = 1 + \mu\frac{\partial}{\partial x} - \frac{c}{2}\int_{-1}^{1} d\mu'(1 + s_1\,\mu\mu'), \qquad (2)$$

where s_1 is the first-order scattering coefficient, which is equal to three times the mean cosine of the scattering angle in the laboratory system. The homogeneous transport equation with linear anisotropic scattering may then be written as

$$L\psi(x,\mu) = (1 + \mu\frac{\partial}{\partial x})\psi(x,\mu)$$

$$- \frac{c}{2}\int_{-1}^{1} d\mu'(1 + s_1\,\mu\mu')\psi(x,\mu') = 0. \qquad (3)$$

We seek solutions to Equation (3) of the form

$$\psi_\nu(x,\mu) = e^{-x/\nu}\ \ \phi_\nu(\mu) \qquad (4)$$

where, for the case $\nu \notin (-1,1)$, the discrete eigenfunctions are

$$\phi_\nu(\mu) = \frac{c\nu}{2(\nu - \mu)}\left[1 + (1-c)s_1\,\mu\nu\right]. \qquad (5)$$

Operating on Equation (4) with L, one obtains

$$L\psi_\nu(x,\mu) = \frac{c}{2}\,e^{-x/\nu}\ (1 + s_1\mu\nu)\ \Lambda(\nu) \qquad (6)$$

where

$$\Lambda(\nu) = 1 - c\nu\tanh^{-1}\frac{1}{\nu} - s_1 c(1-c)\nu^2(\nu\tanh^{-1}\frac{1}{\nu} - 1). \qquad (7)$$

Since $\Lambda(\nu) = 0$ is even in ν, the roots occur in pairs. Let $\xi = \frac{1}{\nu}$, and call the root having the smallest magnitude ξ_0. Since $\Lambda(\xi_0) = 0$, the solution corresponding to $\xi = \xi_0$ is the eigenfunction

$$\psi_0 = \frac{c}{2}\,e^{-x\xi_0}\ \frac{1}{1 - \mu\xi_0}\left[1 + (1-c)\frac{s_1\mu}{\xi_0}\right]. \qquad (8)$$

We note that for $|1 - c| \ll 1$, the root $\xi_0(c)$ to $\Lambda(\xi) = 0$ is given by $\frac{\xi_0^2}{(3 - s_1)(1-c)} = 1 + 0(1-c)$. Hence,

$$\lim_{c\to 1}\ \frac{\xi_0^2}{(3 - s_1)(1 - c)} = 1. \qquad (9)$$

$\xi_0 = 0$ is therefore a double root for $c = 1$.

The eigenfunction corresponding to this root is given by Equation (8) in the limit as $c \to 1$: $\psi_0 = \frac{1}{2}$, since $\xi_0 \to 0$ as $c \to 1$ in the sense given by Equation (9).

[1] K. M. CASE, *Ann. Phys.* 9, 1 (1960).
[2] J. R. MIKA, *Nucl. Sci. Eng.* 11, 415 (1961).