TABLE I

Excursions after Step Reactivity Insertions

Starting at Shut Down			Starting at Operating Power and Temperature			Relative Difference in T_{max}
ρ_0 (β=0.0074)	n_{max} (kW)	T_{max}[a] (°C)	n_{max}	$\frac{n_{max}}{n_0}$	T_{max}[b]	%
10^{-2}	6 760	20	6 960	33.8	25.7	22.5
1.5×10^{-2}	57 800	30	58 000	289	34.3	12.5
2.0×10^{-2}	159 000	40	159 200	795	43.4	7.8

[a]Increase over room temperature.
[b]Increase over operating temperature.

kW-sec(deg C)$^{-1}$ (heat capacity) and n_0 = 200 kW. It must be recalled[1] that when $B = \rho_0 - \beta > 0$ is not large enough, the peak power calculated using Eq. (5) is no longer adequate. However, the final (maximum) temperature reached in the excursion is always given by Eq. (4), and the difference in the values of T_{max} calculated using Eqs. (4) and (4a) becomes very large.

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Comparisons of Exact and S_N Solutions of the Monoenergetic Critical Equation with Anisotropic Scattering

One of the useful applications of the method of singular integral equations[1,2,3] is to provide exact solutions of the Boltzmann transport equation against which numerical code solutions can be compared. We have made such a comparison by calculating the critical thickness of a plane homogeneous slab in the monoenergetic approximation, considering the transport equation

$$\mu \frac{\partial \Psi(x,\mu)}{\partial x} + \Sigma_t \Psi(x,\mu) = \left(\frac{\Sigma_s^{iso} + \nu\Sigma_f}{2}\right) \int_{-1}^{1} \Psi(x,\mu')d\mu' + \Sigma_s^{aniso} \sum_{n=0}^{2} \frac{2n+1}{2} b_n P_n(\mu) \int_{-1}^{1} P_n(\mu')\Psi(x,\mu')d\mu', \tag{1}$$

subject to the boundary conditions

$$\Psi(t/2,\mu) = 0 \qquad \mu < 0$$
$$\Psi(-t/2,\mu) = 0 \qquad \mu > 0. \tag{2}$$

In Eq. (1) the anisotropic scatterer is represented by a three-term Legendre expansion in a form that maintains the anisotropic-scattering cross section as a separate free parameter. Truncation of the expansion is justified either on the grounds that the expansion coefficients become small or on the grounds that the angular flux is relatively isotropic. We have solved Eqs. (1) and (2) for several values of the secondaries ratio $c + c'$ where c is the anisotropic-scattering ratio

$$c = \Sigma_s^{aniso}/\Sigma_t \tag{3}$$

and c' is the isotropic secondaries ratio

$$c' = (\Sigma_s^{iso} + \nu\Sigma_f)/\Sigma_t . \tag{4}$$

For each value of $c + c'$ we have varied c to observe the effect of increasing anisotropy of scattering. For the purpose of calculation the b_n of elastic hydrogen scattering were used ($b_0 = 1$, $b_1 = \frac{2}{3}$, $b_2 = \frac{1}{4}$), but no restriction on the calculations is implied as long as the truncation is justified. For elastic hydrogen scattering $b_3 = 0$ and $b_4 = -\frac{1}{24}$. For these coefficients no appreciable

[1]A. LEONARD and T. W. MULLIKIN, *J. Math. Phys.*, **5**, 399, (1964).
[2]G. J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," Argonne National Laboratory Report ANL-6787, (1963).
[3]K. M. CASE, *Ann. Phys.*, (N. Y.) **9**, 1, (1960).

difference between three- and five-term Legendre expansions is observed when discrete eigenvalues are computed[4]. However, the scattering is anisotropic enough to give different calculated values of the critical half-thickness when a P_1 or two-term and a P_2 or three-term scattering expansion is used.

Exact solutions were obtained by using the results of Ref. 1 in which the critical problem for an anisotropically scattering slab is discussed. For detailed results and definitions of the following, the reader is referred to this paper.

As it was shown there explicitly for isotropic and linearly anisotropic scattering, one can reduce the criticality search for the present problem to the following iteration procedure for the critical thickness, τ:

$$\tau_n = \pi |\nu_0| - 2z_0 - R(\tau_{n-1}) \qquad n = 0,1, \ldots, \quad R(\tau_{-1}) \equiv 0, \tag{5}$$

where

- ν_0 is the discrete eigenvalue of the Boltzmann equation
- z_0 is the Milne extrapolation distance appropriate to anisotropic scattering
- R is a correction term;

in addition to terms which are τ-independent, R contains functionals of two functions $g^{\pm}(\sigma,\tau)$ which satisfy the Fredholm equations

$$g^{+}(\sigma,\tau) = + \exp(-\tau/\sigma) \int_0^1 K(\sigma,\nu) g^{+}(\nu,\tau) d\nu + \sigma N(-\sigma) \exp(-\tau/\sigma) \tag{6}$$

$$g^{-}(\sigma,\tau) = - \exp(-\tau/\sigma) \int_0^1 K(\sigma,\nu) g^{-}(\nu,\tau) d\nu + \sigma N(-\sigma) \exp(-\tau/\sigma), \tag{7}$$

where K has uniform norm $(= \max_{\{0\le\sigma\le1\}} \int_0^1 |K(\sigma,\nu)| d\nu)$ less than one so that Eqs. (6) and (7) may be solved by iteration with a maximum error of $0(e^{-N\tau})$ in the N'th iteration. For each iteration indicated by Eq. (5), iterative solutions of Eqs. (6) and (7) are therefore required.

Since ν_0 is the zero of a certain transcendental function, depending upon the order of anisotropic scattering[4], a simple zero-finder routine with a convergence criterion of 10^{-8} was used. The extrapolation distance, z_0, and other τ-independent constants appearing in R, however, are composed of integrals over $[0,1]$. An extremely accurate Adams-Moulton integration scheme was used for these calculations. A truncation error of less than 10^{-8} at each step was specified. The equations for $g^{\pm}$ were iterated using fifth-order Romberg[5] integration with 129 grid points. The iteration of these equations was terminated after the maximum absolute difference between two consecutive iterates was less than 10^{-8}. The only remaining computation in which extreme care might have been required was in the calculation of the 129 tabular values of the function $N(-\sigma)$, each one requiring an integration over $[0,1]$. At first the Adams-Moulton scheme was used to obtain high accuracy. Subsequently, it was found that Romberg integration was sufficiently accurate since $N(-\sigma)$ arises only in the computation of $g^{\pm}$, which in turn appear only in the correction term $R(\tau)$.

Finally, the iteration indicated by Eq. (5) was continued until two consecutive values of τ differed by less than 10^{-6}.

Comparison solutions were obtained using the DTF transport code[6], modified to allow up to ten-term Legendre scattering expansions in slabs or spheres. The DTF code is a discrete-ordinates code in which the boundary condition (2) is replaced by

$$\Psi(t/2,\mu_j) = 0 \qquad \mu_j < 0$$
$$\Psi(-t/2,\mu_j) = 0 \qquad \mu_j > 0, \tag{8}$$

where μ_j are the discrete directions chosen for the angular quadrature of Eq. (1). A DP_7 quadrature was used and in the actual calculation a reflective-center boundary condition

$$\Psi(0, \mu_{n+1-j}) = \Psi(0, \mu_j) \qquad j = 1,2,\ldots, n/2 \tag{9}$$

was used. Here $n = 16$ and the 16 μ_j are the abscissae for DP_7 quadrature on $[-1,1]$. Seventy-five spatial intervals were used on the interval $[0,t/2]$, with the first 73 intervals equal and the last two spaced by $x_{74} = 4.95$, $x_{75} = 4.99$ and $x_{76} = 5.00$ (with $t/2 = 5.00$). The DTF code searches for the critical half-thickness by computing k for a succession of outer boundaries starting from a guessed value and a first modification. A convergence criterion on the multiplication, k, (and on the spatial flux) of 10^{-6} was used.

The Legendre polynomials are generated recursively by the code and evaluated at discrete $\mu = \mu_j$. Legendre moments of the flux are computed by quadrature

[4] K. D. LATHROP, "Anisotropic Scattering Approximations in the Boltzmann Transport Equation," Los Alamos Scientific Laboratory Report LA-3051, (1964).

[5] E. L. STIEFEL, *An Introduction to Numerical Mathematics*, Academic Press, New York, (1963), p. 149.

[6] B. G. CARLSON, W. J. WORLTON, W. GUBER and M. SHAPIRO, "DTF Users Manual," United Nuclear Corporation Report UNC Phys/Math-3321 (Two Volumes),(1963).

$$\frac{2n+1}{2}\int_{-1}^{1} P_n(\mu)\Psi(x,\mu)d\mu = \frac{2n+1}{2}\sum_{j=1}^{16} w_j P_n(\mu_j)\Psi(x,\mu_j), \qquad (10)$$

where the w_j are the weights associated with the quadrature, here DP_7. Since the P_n are polynomials, integrals of the applicable P_n themselves are exact (neglecting round-off) with DP_7 quadrature.

For the actual cross sections entered in the code it was assumed that $\Sigma_t = 1.0$, that all absorption was due to Σ_f and that $\nu = 2.5$. Then

$$\begin{aligned} \Sigma_a &= 2(c + c' - 1)/3 \\ \nu\Sigma_f &= 5(c + c' - 1)/3 \\ \Sigma_t &= 1.0 \\ \Sigma_s^{tot} &= [5 - 2(c + c')]/3, \end{aligned} \qquad (11)$$

and the anisotropic components are given by cb_1 and cb_2.

The results of the exact and DTF calculations are compared in Table I. In all cases the agreement is so good that an analysis of the difference is probably not meaningful. Note, however, that for two P_2 cases, $c + c' = 1.3$ and 1.4 with $c = 0.9$, the exact and end-point values are not given. For these cases there are two discrete eigenvalues of the Boltzmann equation, leading to additional complexities in the computation which cannot be handled by the present program used for exact solution. The table also includes the value, used as the first approximation in the exact calculation (see Eq. (5)), from the end-point formula

$$\tau/2 = \pi|\nu_0|/2 - z_0 . \qquad (12)$$

The agreement of these values with the exact values establishes the accuracy of the end-point approximation for anisotropic scattering.

The corresponding values of the Milne extrapo-

TABLE I

Slab Critical Half-Thickness (mean free path)

c	P_1(DTF)	P_1(Exact)	P_1(End-point)	P_2(DTF)	P_2(Exact)	P_2(End-point)
			$c + c' = 1.05$			
0.1	3.39225	3.39216	3.39216	3.39042	3.39032	3.39032
0.3	3.60356	3.60346	3.60346	3.59724	3.59714	3.59714
0.5	3.86368	3.86358	3.86358	3.85135	3.85126	3.85126
0.7	4.19568	4.19558	4.19558	4.17506	4.17495	4.17495
0.9	4.64212	4.64203	4.64203	4.60935	4.60927	4.60927
			$c + c' = 1.1$			
0.1	2.16524	2.16519	2.16525	2.16311	2.16306	2.16313
0.3	2.28290	2.28285	2.28291	2.27558	2.27553	2.27559
0.5	2.42452	2.42447	2.42452	2.41031	2.41026	2.41033
0.7	2.59999	2.59994	2.59998	2.57637	2.57632	2.57639
0.9	2.82629	2.82625	2.82628	2.78923	2.78917	2.78924
			$c + c' = 1.2$			
0.1	1.31521	1.31519	1.31567	1.31299	1.31296	1.31347
0.3	1.37256	1.37254	1.37308	1.36498	1.36496	1.36558
0.5	1.43939	1.43935	1.43997	1.42486	1.42484	1.42561
0.7	1.51884	1.51882	1.51948	1.49506	1.49505	1.49604
0.9	1.61576	1.61575	1.61646	1.57930	1.57929	1.58061
			$c + c' = 1.3$			
0.1	0.95354	0.95352	0.95473	0.95142	0.95140	0.95266
0.3	0.98811	0.98810	0.98955	0.98098	0.98096	0.98262
0.5	1.02747	1.02746	1.02922	1.01398	1.01399	1.01621
0.7	1.07292	1.07291	1.07507	1.05123	1.05127	1.05439
0.9	1.12636	1.12635	1.12903	1.09382	----[a]	----[a]
			$c + c' = 1.4$			
0.1	0.74726	0.74727	0.74930	0.74532	0.74529	0.74743
0.3	0.77031	0.77032	0.77288	0.76381	0.76378	0.76669
0.5	0.79607	0.79606	0.79936	0.78393	0.78396	0.78800
0.7	0.82519	0.82519	0.82948	0.80595	0.80610	0.81199
0.9	0.85853	0.85853	0.86424	0.83022	----[a]	----[a]

[a] Not calculable by present exact program due to the appearance of a second discrete eigenvalue.

lation distance, z_0, as calculated from the exact analysis are given in Table II.

We feel that the type of comparison made above is important in verifying the accuracy of any numerical code that solves the transport equation. For the DTF code, the handling of the boundary conditions, the eigenvalue search, and the linear and P_2 scattering options is apparently accurate. Although no information is obtained about material spatial variation or multigroup treatment, such comparisons can be made, albeit with very complicated exact solutions. The table provides solutions against which other codes can be compared, and in addition contains useful information about the behavior of critical thickness for anisotropic scattering. Even for the simple monoenergetic, homogeneous case, additional meaningful comparisons can be made. Exact solutions for critical radii for one-dimensional spheres can be obtained with relatively minor changes in the slab critical equation, and such solutions could be used to examine the treatment of ray-to-ray transfers (streaming) in curved geometry. Mitsis[2] has given an exact critical equation for cylindrical geometry, solutions to which could be used to investigate the accuracy of two-dimensional angular quadrature. Although, in themselves, such comparisons verify only parts of ofttimes extremely complex codes, they provide the foundation upon which confident numerical computing can be based.

TABLE II

Extrapolation Distance, z_0 (mfp)

$c + c'$	c	$(z_0)_{P_1}$	$(z_0)_{P_2}$
1.05	0.1	0.72414	0.72394
	0.3	0.84222	0.84151
	0.5	1.00594	1.00456
	0.7	1.24815	1.24580
	0.9	1.64322	1.63928
1.1	0.1	0.69042	0.68982
	0.3	0.80043	0.79826
	0.5	0.95181	0.94735
	0.7	1.17336	1.16537
	0.9	1.52899	1.51498
1.2	0.1	0.63139	0.63013
	0.3	0.72719	0.72252
	0.5	0.85747	0.84753
	0.7	1.04518	1.02674
	0.9	1.34000	1.30673
1.3	0.1	0.58146	0.57974
	0.3	0.66524	0.65872
	0.5	0.77811	0.76395
	0.7	0.93889	0.91204
	0.9	1.18783	...[a]
1.4	0.1	0.53873	0.53670
	0.3	0.61232	0.60452
	0.5	0.71061	0.69349
	0.7	0.84928	0.81635
	0.9	1.06161	...[a]

[a]Not calculable by present exact program due to the appearance of a second discrete eigenvalue.

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A Note on the Inverse Kinetics Analysis

In their article[1] Murray, Bingham and Martin have presented some solutions of the reactor kinetics equations for the reactivity variation required to achieve specified power responses. They show the usefulness of such an inverse method and demonstrate it by several examples. It is felt that an important case could be added to the list of power functions considered in the above-mentioned paper.

For the reactor power function of the type

$$n = n_0 \exp(\alpha t)\,(1 + A \sin\omega t)$$

the corresponding reactivity function can be found by solving reactor point kinetics equations. This is

$$\rho = \alpha\left(\ell^* + \sum_i \frac{\beta_i}{\lambda_i + \alpha}\right) + \sum_i \frac{\beta_i \lambda_i \omega^2}{(\lambda_i + \alpha)^3 + \omega^2(\lambda_i + \alpha)} \times$$

$$\times \frac{A \sin\omega t}{1 + A \sin\omega t} + \left[\omega \ell^* + \sum_i \frac{\beta_i \lambda_i \omega}{(\lambda_i + \alpha)^2 + \omega^2}\right] \times$$

$$\times \frac{A \cos\omega t}{1 + A \sin\omega t} - \frac{Q\ell^* \exp(-\alpha t)}{n_0\,(1 + A \sin\omega t)} -$$

$$- \sum_i \left[\frac{\lambda_i \ell^* C_{i0}}{n_0} - \frac{\beta_i \lambda_i}{\alpha + \lambda_i} + \frac{A \beta_i \lambda_i \omega}{(\lambda_i + \alpha)^2 + \omega^2}\right] \times$$

$$\times \frac{\exp(-\alpha t - \lambda_i t)}{1 + A \sin\omega t},$$