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Letters to the Editors

Comments on the Existence of Exceptional Frequencies in Multiplying Media

Moore[1] has examined the dispersion of thermal neutrons in a modulated bare multiplying medium within the framework of age-diffusion theory. In particular, the possibility of exceptional frequencies is noted, i.e. pass frequencies, which permit no attenuation of the source signal, and stop frequencies, which permit no spatial harmonic dependence. By examining the system corresponding to modulation with zero-frequency neutrons, Moore is able to conclude that for a critical system, the only non-zero exceptional frequencies are stop frequencies.

As a footnote to this work, two further results can be obtained from this model: 1) For thermal reactor systems characterized by a thermal-diffusion lifetime at least as large as the slowing-down time, all non-zero exceptional frequencies are stop frequencies. This applies to finite systems and to all physically realizable values of the infinite multiplication factor. 2) There exists a minimum non-zero stop frequency which is related to the slowing-down time by the condition

$$\tilde{\omega}_{\min} > \frac{\pi}{\ell_r}, \tag{1}$$

where

$$\ell_r = \int_0^u \frac{du'}{\xi v \Sigma_t} \tag{2}$$

is the slowing-down time.

Starting with the two simultaneous equations obtained from the real and imaginary parts of the thermal-neutron dispersion law (Eq. (37) of Ref. 1), restricting to non-zero exceptional frequencies (B=0), and eliminating terms involving $(L^2A + 1)$ results in

$$e^{-A\tau} = -\frac{\tilde{\omega}\ell_s/k_\infty}{k_2\cos\tilde{\omega}\ell_r + k_1\sin\tilde{\omega}\ell_r}, \tag{3}$$

or

$$A = -\frac{1}{\tau}\ln\left[-\frac{\tilde{\omega}\ell_s/k_\infty}{k_2\cos\tilde{\omega}\ell_r + k_1\sin\tilde{\omega}\ell_r}\right]. \tag{4}$$

For the argument of the logarithm to be positive, one requires

$$k_2\cos\tilde{\omega}\ell_r + k_1\sin\tilde{\omega}\ell_r < 0. \tag{5}$$

The delayed-neutron functions, $k_1(\tilde{\omega})$ and $k_2(\tilde{\omega})$, are positive for all $\tilde{\omega}$.

Requirement (5) immediately restricts the trigonometric argument from lying in the first quadrant, i.e.

$$\tilde{\omega}\ell_r > \frac{\pi}{2}. \tag{6}$$

Since slowing-down times, ℓ_r, are typically of order 10^{-4} - 10^{-5} sec, $\tilde{\omega}$ is restricted to frequencies of the order 10^4 - 10^5 sec^{-1}. But for large $\tilde{\omega}$,

$$\begin{aligned} k_1(\tilde{\omega}) &\to 1 - \beta \approx 1 \\ k_2(\tilde{\omega}) &\to \frac{\Sigma_i\lambda_i\beta_i}{\tilde{\omega}} \to 0. \end{aligned} \tag{7}$$

(Typically, for U^{235} and $\tilde{\omega}=\frac{\pi}{2\ell_r}$ = 2500 cycles/sec (corresponding to $\ell_r = 10^{-4}$ sec) $k_2 = 1.75 \times 10^{-7}$.) Since $k_2(\tilde{\omega}) \ll k_1(\tilde{\omega})$ at these frequencies, condition (5) cannot be satisfied in the second quadrant with the possible exception of values of $\tilde{\omega}\ell_r$ very close to π. In this range

$$\cos\tilde{\omega}\ell_r \approx -1$$

$$\sin\tilde{\omega}\ell_r = \sin(\pi-\tilde{\omega}\ell_r) \approx \pi-\tilde{\omega}\ell_r$$

so that requirement (5) becomes

$$-k_2 + k_1(\pi-\tilde{\omega}\ell_r) < 0$$

or

$$\tilde{\omega}\ell_r > \pi - \frac{k_2}{k_1} \approx \pi,$$

which is inequality (1).

Further, requirement (5) is satisfied only for

$$\pi \leqslant \tilde{\omega}\ell_r < 2\pi \tag{8}$$

or even multiples of these limits. The value $\tilde{\omega}\ell_r = 2\pi$ fails to satisfy requirement (5) and must be excluded.

[1] M. N. MOORE, "The Determination of Reactor Dispersion Laws from Modulated Neutron Experiments," this issue, p. 565.

The fact that the exceptional frequencies are stop frequencies is demonstrated by examining the magnitude of A. Referring to Eq. (4) above, if the argument of the logarithm exceeds unity, A is negative and the exceptional frequencies correspond to stop frequencies, i.e. $A < 0$, if

$$\left| \frac{\tilde{\omega} \ell_s / k_\infty}{k_2 \cos \tilde{\omega} \ell_\tau + k_1 \sin \tilde{\omega} \ell_\tau} \right| > 1$$

or

$$\tilde{\omega} \ell_s / k_\infty > |k_2 \cos \tilde{\omega} \ell_\tau + k_1 \sin \tilde{\omega} \ell_\tau|. \quad (9)$$

The value of the right hand side of expression (9) is bounded by zero and unity. Incorporation of the upper bound results in

$$\tilde{\omega} > \frac{k_\infty}{\ell_s},$$

which satisfies expression (1) for systems whose thermal-diffusion lifetime is at least as large as the slowing-down time.

Similar arguments for $A > 0$ lead to the condition

$$\tilde{\omega} < \frac{k_\infty}{\ell_s},$$

which violates expression (1). Thus, for bare thermal multiplying media (and within the framework of the-age diffusion model) the only exceptional frequencies are stop frequencies.

The simplicity of requirement (1) is a consequence of the continuous slowing-down model employed. It is interesting and perhaps surprising that the characteristic time is the slowing-down time and not, for example, the mean time to fission. It should be emphasized, too, that the simple age-diffusion model is not necessarily valid at the high frequencies contemplated. This aspect is currently being investigated by examining the dispersion relationships obtained from a telegrapher age-diffusion model.

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Multilevel Cross Sections and Reactor Criticality*

Resonance cross sections of the fissile isotopes have been treated in two essentially distinct fashions in reactor analysis. In the thermal energy range, which is normally chosen to extend from 0 to about 0.5 eV, these cross sections may be represented as empirical pointwise functions of energy. In the epithermal energy range, on the other hand, resonances are so numerous and so sharp that one must represent them analytically. Since the Breit-Wigner single-level formula[1] has been universally used in this connection, it becomes interesting to examine the effects that the approximate nature of this formula may have on criticality calculations.

One possible defect of the single-level formula is made evident by considering the fission-to-absorption ratio in a pure U^{235} fuel element. Near an epithermal resonance energy, E_λ, the fission and absorption of the uranium are dominated by the resonance in question, and the fission-to-absorption ratio of the fuel predicted by the single-level formula is, to a good approximation, the ratio of the partial widths for fission and absorption, $\Gamma_f^\lambda/(\Gamma_f^\lambda + \Gamma_\gamma^\lambda)$, a quantity independent of fuel-plate thickness or shape. The general Wigner-Eisenbud theory[2,3], on the other hand, states that

$$E^{1/2}\sigma_c = \text{const} \sum_c \left| \sum_{\lambda,\lambda'} (\Gamma_{n,o}^\lambda)^{1/2} (\Gamma_c^{\lambda'})^{1/2} A_{\lambda\lambda'} \right|^2, \quad (1)$$

where

c refers to either fission or capture channels

$(A^{-1})_{\lambda\lambda'} = (E_\lambda - E)\delta_{\lambda\lambda'} - \frac{1}{2}\, i \sum_c (\Gamma_c^\lambda)^{1/2} (\Gamma_c^{\lambda'})^{1/2}$

$\Gamma_{n,o}^\lambda$ is the reduced neutron width of the resonance at E_λ.

In agreement[3] with experiment, Eq. (1), which we shall refer to as the multilevel formula, predicts that while the capture cross section σ_γ is symmetric about E_λ, the fission cross section is not. The fission-to-absorption ratio in a fuel element is thus a function of its dimensions. Since to a first approximation the effective multiplication constant of a reactor is porportional to $1/(1+\alpha)$, it becomes necessary to investigate the variation of this ratio with fuel plate geometry.

Multilevel values of $1/(1+\alpha)$ (averaged over the energy range between 0.625 and 3.7 eV) were calculated for a series of slab lattices composed of U^{235} fuel plates (10% by atom U^{235} in Zr) and water at 293°K. The Doppler-broadened multilevel cross

*Work performed under the auspices of the USAEC.

[1] A. M. WEINBERG and E. P. WIGNER, *The Physical Theory of Neutron Chain Reactors*, University of Chicago Press, Chicago, Illinois (1958).

[2] E. P. WIGNER and L. EISENBUD, *Phys. Rev.*, **72**, 29 (1947).

[3] E. VOGT, *Phys. Rev.*, **112**, 203 (1958), and *Phys. Rev.*, **118**, 724 (1960).