NUCLEAR SCIENCE AND ENGINEERING: 28, 155-158 (1967)

Letters to the Editor

Further Comment on the Optimal Shutdown Control

In the Letters to the Editor[1,2], Ash replied to the question of Roberts and Smith by writing "the time optimal extremal is equivalent to the minimax extremal where the minimum time coincides with the allowable shutdown time of the minimax solution. However, the converse is not true. That is, minimax extremals are not necessarily time optimal extremals." It can be shown, however that this last statement is not true as far as the xenon minimax and the time optimal problems are concerned. The xenon minimax problem can be converted to the time optimal one[3] and the time optimal solutions have been obtained[3,4]. Roberts and Smith presented the equivalence of these two problems[5], but their explanation seems to be inadequate.

The equivalence of these two problems can be verified easily in the following way. First, it is clear that in the xenon-iodine phase plane, the sections of target curves which are meaningful lie in the region between the maximum line M and the equilibrium curve E and that the target curve moves to the left with decreasing xenon maximum value, as shown in Fig. 1. It also is evident that a finite time is necessary for the phase point on a certain curve to move onto another curve which is separated by a finite distance, and that the initial phase point should lie in the right side of the target curve.

For the proof of the equivalence of the two problems, it must be shown that the time optimal trajectory (the minimum time = T) between the point A and the target curve C corresponding to the xenon maximum $x = x_m$ gives the minimum of x when the allowed shutdown time, or transition time is T and that, conversely, the minimax trajectory (the minimum of the xenon maximum = x_m) for the allowed shutdown time T, is also time optimal one between A and C.

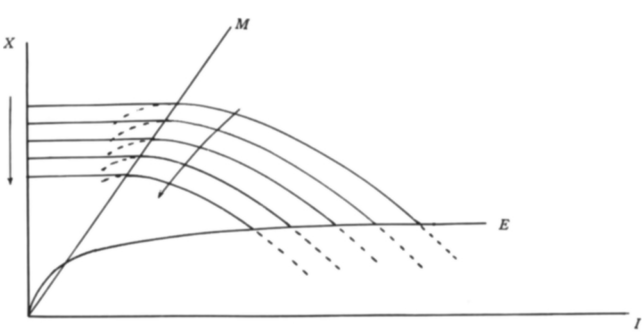

Fig. 1. Family of target curves.

Assume that a time-optimal trajectory V, whose transition time is T, has been found between the initial point A and the target curve C corresponding to $x = x_m$, as shown in Fig. 2. In reality, the optimal trajectories are not of such form, but the argument does not lose its generality. Now, if there exists a trajectory V' which gives a xenon maximum x_m' smaller than x_m and whose transition time is T, this trajectory must cross the curve C to reach the curve C' corresponding to x_m' ($< x_m$). Let the transition time from A to the cross point K be T_1' and that from K to B' be T_2', then $T = T_1' + T_2'$. Since $T_2' > 0$, so $T_1' < T$. This contradicts the assumption of the time optimality of the trajectory V between A and C. Thus, this time-optimal trajectory gives the minimum of the xenon maximum when the transition time T is fixed.

Now assume that a minimax trajectory W has been found for the transition time T (Fig. 3), and that the corresponding xenon maximum turned out to be x_m. If there exists a

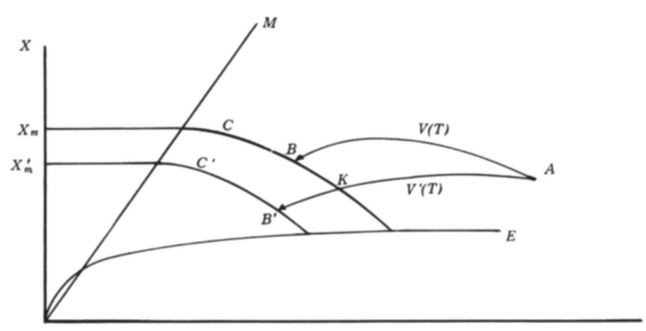

Fig. 2. Time optimal trajectory is also minimax trajectory.

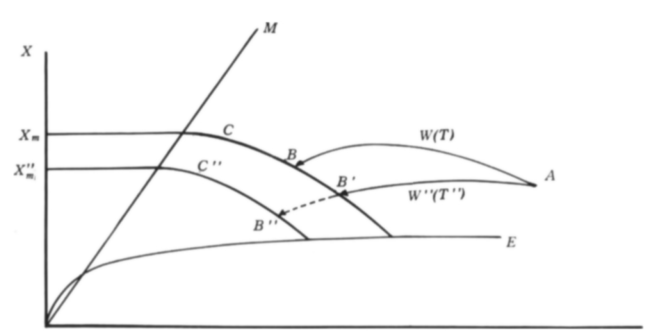

Fig. 3. Minimax trajectory is also time optimal one.

[1] J. J. ROBERTS and H. P. SMITH, Jr., *Nucl. Sci. Eng.*, 25, 212 (1966).
[2] M. ASH, *ibid.*, 25, 213 (1966).
[3] Y. SHINOHARA and J. VALAT, *C. R. Acad. Sci. Paris*, 259, 1623 (1965).
[4] J. J. ROBERTS and H. P. SMITH, Jr., *Nucl. Sci. Eng.*, 22, 470 (1965).
[5] J. J. ROBERTS and H. P. SMITH, Jr., *ibid.*, 23, 397 (1965).

trajectory W'' which reaches the curve C in a time T'' shorter than T, then it can go further for the duration of time $T - T''$ (> 0), reaching a curve C'' which may give a xenon maximum smaller than x_m. This contradicts the assumption of minimax trajectory W, that is to say, the minimax trajectory W is also the time-optimal trajectory between A and C. Thus the equivalence of the xenon minimax and the time-optimal problems have been proved.

The multipulse solutions[6] seem, by our opinion, to have resulted from the practical computational difficulties sometimes associated with the method of dynamic programming.

As a numerical reference, the recent study of Motoda, Togo, and Oyama[7] shows that the results obtained by time optimal and minimax criteria are identical.

Yoshikuni Shinohara

Japan Atomic Energy Research Institute
Tokai-mura, Ibaraki-ken, Japan

Jean Valat

Centre d'Études Nucléaire de Saclay
B. P. N°2
Gif-sur-Yvette, France

November 15, 1966

[6]M. ASH, *ibid.*, **24**, 77 (1966).
[7]H. MOTODA, Y. TOGO, and A. OYAMA, Preprints for the Joint Meeting in Reactor Physics and Reactor Engineering, Atomic Energy Society of Japan. Fuse (1966).

Optimal Shutdown Control

My comments on "Further Comments . . ." by Y. Shinohara and J. Valat, (this issue) to my earlier rejoinder *Nuclear Science and Engineering* **25**, 213 (1966) to comments by J. J. Roberts and H. P. Smith *Nuclear Science and Engineering* **25**, 212 (1966) on my original article *Nuclear Science and Engineering* **24**, 77-86 (1966) are, after catching my breath and wondering about the philosophical implications of more and more about less and less, the following:

I reiterate my earlier rejoinder that "the time optimal extremal is equivalent to the minimax extremal where the minimum time coincides with the allowable shutdown time of the minimax solution." So far so good. Continuing, "However, the converse is not true. That is, minimax extremals are not necessarily time optimal extremals." Note, in this last statement I am talking about minimax extremals in general, not necessarily those of fixed allowable shutdown time.

Hence, as also discussed further in Chapters 8 and 9 of my monograph *Optimal Shutdown Control in Nuclear Reactors*, Academic Press (1966), one can, for example, multipulse the reactor to keep within the xenon constraint while maintaining the system on a minimax xenon, *but not minimum time*, extremal. After a number of pulses, whose characteristics are determined by the particular system parameters, one will reach the xenon-iodine phase space target curve.

Whether or not the Roberts and Smith explanation of the equivalence of minimax and time-optimal extremals is adequate, as questioned in the above Y. Shinohara, J. Valat Letter, seems to me to be a matter of taste. I think it is.

Multipulse solutions will sometimes result from the computational vagaries of using the dynamic programming method, if one is not careful. This can come about from adding an artificial cost, $10(x/x_c)^{20}$ for example, to the criterion functional to definitely assure that $x = x_c$, the xenon constraint, will not be exceeded by the phase-space extremal trajectory. There are however, other more efficient numerical devices that will accomplish the same thing. This is a negligible price to pay for employing the straight forward method of dynamic programming for this class of problems. Dynamic Programming gets one out of the bind of having to solve a messy two-point boundary value problem (especially on digital machinery), an unfortunate concomitant of the corresponding Pontryagin maximum principle formulation.

Dr. Milton Ash

E. H. Plesset Associates, Inc.
2444 Wilshire Blvd.
Santa Monica, California 90403

November 25, 1966

Comment on the Optimal Shutdown Control

It is a pleasure to reply to "Further Comment on the Optimal Shutdown Control," by Shinohara and Valat[1] in which they correctly note the error in Ash's statement[2] that "minimax extremals are not necessarily time optimal extremals." The proof by Shinohara and Valat, that time optimal extremals are coincident with minimax extremals in which the fixed period of operation corresponds to the minimum time, is the same as that given by Roberts and Smith[3]. The reverse proof, i.e., that minimax extremals are equivalent to time optimal extremals under the conditions noted, is correct, well presented, and nicely extends our initial approach to the problem.

John J. Roberts

Argonne National Laboratory
Argonne, Illinois 60440

Harold P. Smith, Jr.

Department of Nuclear Engineering
University of California
Berkeley, California

January 6, 1967

[1]Y. SHINOHARA and JEAN VALAT, *Nucl. Sci. Eng.*, **27**, 156 (1967).
[2]M. ASH, *Nucl. Sci. Eng.*, **25**, 213 (1966).
[3]JOHN J. ROBERTS and HAROLD P. SMITH, Jr., *Nucl. Sci. Eng.*, **23**, 397 (1965).

Comments on the Time Optimal Xenon Shutdown Problem

An excellent solution of this problem has recently been presented by Roberts and Smith[1]. There is, however, one small point in their analysis which needs clarification. On page 476, the trajectory ABC is considered as a possible time-optimal trajectory in the restricted state space. It is assumed that at $B = x(\tau)$, the flux did not switch in the unrestricted space solution. B is a junction point and the jump conditions of Theorem 3 must be satisfied. That is, p_1 and H must be continuous, but p_2 may jump.

[1]J. J. ROBERTS and H. P. SMITH, Jr., *Nucl. Sci. Eng.*, **22**, 470 (1965).