Letters to the Editor

Generalized Perturbation Formula Versus Classic Perturbation Formula

A generalized first order perturbation formula for the diffusion equation was derived recently.[1] An analysis of this new general formulation has been made to assess the suitability of its replacing the classic formula as suggested.[1] The purpose of this Letter is to bring our results to the notice of the reactor physics community.

In order to understand, rigorously, the relation of the new formulation to the exact solution, an analysis with reference to the specific example of Ref. 1 has been made and is given below.

The unperturbed problem[1] is given by

$$\frac{d^2\phi_0}{dx^2} + \Lambda_0\phi_0 = 0 \ , \quad 0 < x < 1 \tag{1a}$$

and

$$\frac{d^2\phi_0}{dx^2} = 0 \ , \quad 1 < x < T \tag{1b}$$

$$\frac{d\phi_0(0)}{dx} = \phi_0(1+T) = 0 \ .$$

We consider a perturbed problem little more general than the one given in Ref. 1 as given by

$$\frac{d^2\phi}{dx^2} + \Lambda\phi = 0 \ , \quad 0 < x < 1 \ , \tag{2a}$$

$$(1+\epsilon_m)\frac{d^2\phi}{dx^2} = 0 \ , \quad 1 < x < 1+\epsilon_v \ , \tag{2b}$$

and

$$\frac{d^2\phi}{dx^2} = 0 \ , \quad 1+\epsilon_v < x < 1+T \ , \tag{2c}$$

[1] G. C. POMRANING, *Nucl. Sci. Eng.*, **83**, 72 (1983).

TABLE I

The Relative Error in the Perturbation Estimate of the Eigenvalue*

E	$T = 0.0500$	0.1000	0.3000	1.0000	5.0000	10.0000	20.0000
−0.90	0.7657	1.3152	2.0991	1.6248	0.4862	0.2562	0.1315
	−0.7155	−2.6731	−14.7493	−43.6027	−71.0088	−75.8073	−78.3525
−0.70	0.1475	0.2564	0.4218	0.3296	0.0983	0.0517	0.0265
	−0.0426	−0.1622	−0.9524	−2.9113	−4.7713	−5.0950	−5.2664
−0.50	0.044	0.0772	0.1287	0.1014	0.0301	0.0159	0.0081
	−0.0075	−0.0284	−0.1699	−0.5315	−0.8761	−0.9357	−0.9673
3.00	0.1943	0.3349	0.5712	0.4763	0.1402	0.0727	0.0370
	−0.0039	−0.0143	−0.0847	−0.2838	−0.4904	−0.5256	−0.5439
10.00	0.7827	1.3466	2.2998	1.9599	0.5889	0.3027	0.1521
	−0.0057	−0.0207	−0.1214	−0.4067	−0.7161	−0.7706	−0.7986
15.00	1.2101	2.0812	3.5550	3.0449	0.9263	0.4757	0.2380
	−0.0061	−0.0220	−0.1284	−0.4298	−0.7595	−0.8186	−0.8489
20.00	1.6385	2.8177	4.8135	4.1344	1.2696	0.6525	0.3256
	−0.0062	−0.0226	−0.1322	−0.4420	−0.7823	−0.8440	−0.8758

*Upper entry is the classical formula; lower entry is the generalized formula. Relative error = (exact value − perturbation estimate)/exact value.

where ϵ_m is the change in material property over the volume ϵ_v and

$$\frac{d\phi(0)}{dx} = \phi(1 + T) = 0 \ .$$

One can easily obtain the exact solution for Λ as

$$\sqrt{\Lambda} \tan\sqrt{\Lambda} = \frac{-(1 + \epsilon_m)}{[(1 + \epsilon_m)(\epsilon_v - T) - \epsilon_v]} \ . \tag{3}$$

A perturbation series expansion can be made for Λ around Λ_0, either in terms of ϵ_m or in terms of ϵ_v giving

$$\Lambda = \Lambda_0 + \partial\Lambda/\partial\epsilon_m|_{\epsilon_m=0} + O(\epsilon_m^2) \tag{4a}$$

and

$$\Lambda = \Lambda_0 + \partial\Lambda/\partial\epsilon_v|_{\epsilon_v=0} + O(\epsilon_v^2) \ . \tag{4b}$$

The usual (classic) result using Eq. (8) of Ref. 1 and the generalized perturbation result using Eq. (11) of Ref. 1 for this problem are respectively given by

$$\delta\Lambda_u(\text{classic}) = \frac{2\Lambda_0\epsilon_v\epsilon_m}{1 + T + T^2\Lambda_0} \tag{5a}$$

and

$$\delta\Lambda_g(\text{generalized}) = \frac{2\Lambda_0\epsilon_m\epsilon_v}{(1 + \epsilon_m)(1 + T + T^2\Lambda_0)} \ . \tag{5b}$$

From Eqs. (3), (4), and (5), it can be easily shown that

$$\delta\Lambda_u = \epsilon_m \frac{\partial\Lambda}{\partial\epsilon_m}\bigg|_{\epsilon_m=0} \tag{6a}$$

and

$$\delta\Lambda_g = \epsilon_v \frac{\partial\Lambda}{\partial\epsilon_v}\bigg|_{\epsilon_v=0} \ . \tag{6b}$$

For $\epsilon_v = T$ and $\epsilon_m = E$, we get the results of Ref. 1 and Eqs. (5a) and (5b) reduce to $(\delta\Lambda_u)$ and $(\delta\Lambda_g)$ of Ref. 1, respectively. Thus, with reference to this specific example of Ref. 1, the classic result is accurate to first order in ϵ_m, and the new general formulation of Ref. 1 is accurate to first order in ϵ_v. Qualitatively, it may be possible to say[2] that the classic formula can be better for small ϵ_m, and the general formula (one accurate to first order only in ϵ_v) can be better for small ϵ_v. However, rigorously, the two formulations in this example are related to expansions in different parameters, and their accuracy, depending on both ϵ_m and ϵ_v, cannot be compared. The exact solution can be a complicated function of ϵ_m and ϵ_v, and the range of parameters over which one is more accurate than the other cannot be quantified.

Numerical calculations for a wider range of parameters of the perturbed problem given in Ref. 1 ($\epsilon_m = E$, $\epsilon_v = T$) have been made and are presented in Table I. It can be seen that our theoretical analysis satisfactorily explains the trend in the results for different values of E and T. While the general formulation is more accurate in the left region of the table, the classic formulation is better in the remaining region.

Thus, the above analysis restricted to the specific example, we hope, clearly shows that the generalized formulation of Ref. 1 has an advantage over the classic formulation only in a limited range, and the suggestion[1] that it replace the classic formula "as the fundamental first order diffusion theory perturbation formula" does not appear to be convincing.

J. V. Muralidhar Rao

Reactor Research Centre
310 GSB
Kalpakkam 603 102
India

July 6, 1983

Reply to "Generalized Perturbation Formula Versus Classic Perturbation Formula"

With regard to the recent Letter by Rao,[1] three points need to be made.

1. Rao apparently does not understand the newly derived (general) formula.[2] He states[1] " . . . the classic result is accurate to first order in ϵ_m, and the new general formulation is accurate to first order in ϵ_v." This is incorrect. The correct statement is that the new general formulation is accurate to first order in ϵ_m *and* ϵ_v.

2. By a specific example, Rao shows that the classic formula is more accurate than the new formula for certain values of ϵ_m and ϵ_v (E and T in the example). However, an examination of the table of Rao shows that this is the case when both ϵ_m and ϵ_v are large, of order one or greater. Obviously any first order perturbation theory based on an expansion in a small parameter is likely to fail if this parameter is not small. In the present context, if the two formulas are misapplied, it seems to me that in any given problem there is roughly a 50/50 chance as to which one is more inaccurate.

3. Based on the numerical results referred to above, Rao questioned the suggestion[2] that the new formula replace the classic formula as the fundamental first order perturbation formula. I stand by this suggestion. The new formula, when properly applied (i.e., for small ϵ_m or ϵ_v), clearly treats a larger class of problems. The classic formula is a special case of the new formula, restricted to perturbations first order in ϵ_m alone. It is in this sense that the new formula should replace the classic formula.

G. C. Pomraning

University of California
Los Angeles, California 90024

August 9, 1983

[2]J. V. MURALIDHAR RAO and S. M. LEE, *Nucl. Sci. Eng.*, **84**, 72 (1983).

[1]J. V. MURALIDHAR RAO, *Nucl. Sci. Eng.*, **85**, 432 (1983).
[2]G. C. POMRANING, *Nucl. Sci. Eng.*, **83**, 72 (1983).