A Plausibility Argument by B. Davison for the Completeness of the Elementary Solutions to the One-Dimensional Neutron-Transport Equation*

Recently the author stumbled upon a proof, given by the late B. Davison in a declassified Canadian report[1] MT-112, that the elementary solutions to the steady-state monoenergetic one-dimensional neutron transport equation form a complete set (in a sense defined below)[2]. The proof constitutes Appendix B of this report, which is dated January 31, 1945. A more general and much more useful derivation of this completeness property has since been supplied by K. M. Case[3]. Case's derivation is more useful because it is constructive; i.e., the method by which the expansion coefficients are to be calculated is displayed. Davison's derivation is nevertheless of interest for several reasons. First, it is far more intuitive than Case's derivation, which rests upon the solution to a certain class of singular integral equations; second, so far as the author is aware, it is the earliest published reference to this interesting subject; and finally, the method of proof is of interest for its own sake—as it suggests a novel method by which infinite slab problems may be treated. Because of the general unavailability of MT-112, Davison's proof is presented below in its entirety, with only some of the equation numbers changed. For completeness, we also paraphrase Davison's derivation of the singular solutions, which is found in the body of his report. Davison precedes his derivation with the statement, "This family of eigensolutions has been already discussed by various authors before, but in the absence of any definite record of these discussions it will be more convenient to derive this family of eigensolutions anew."

We write the transport equation in the form

$$\mu \frac{\partial \Psi(z,\mu)}{\partial z} + \Psi(z,\mu) = \frac{1-\alpha}{2}\int_{-1}^{+1} \Psi(z,\mu')\, d\mu' \qquad (1)$$

where $\alpha = 1 - c$, and c is the mean number of secondaries. Trying a separable solution,

$$\Psi(z,\mu) = \Psi_0(z)\, f(\mu), \qquad (2)$$

in which f is to satisfy the normalization condition

$$\int_{-1}^{+1} f(\mu)\, d\mu = 1, \qquad (3)$$

we deduce that

$$\left[\mu \frac{\partial \log \Psi_0(z)}{\partial z} + 1\right] f(\mu) = \frac{1-\alpha}{2}. \qquad (4)$$

Thus,

$$\Psi_0(z) = \text{const} \cdot \exp(\nu z) \qquad (5)$$

(note that Davison's ν is the negative reciprocal of Case's), and

$$f(\mu) = \frac{1-\alpha}{2}\left[(1+\mu\nu)^{-1} + c(\nu)\, \delta(\mu + \nu^{-1})\right]. \qquad (6)$$

Although Eq. (5) is a rigorous result of the theory of distributions[4], it was written by Davison without comment at a time when the theory of distributions was in the process of being developed; presumably in the spirit that when one divides by $\mu + \nu^{-1}$ one can pick up a delta function [because $(\mu + \nu^{-1})\,\delta(\mu + \nu^{-1})$ is zero everywhere]. Moreover, if $(1 + \mu\nu)^{-1}$ is not integrable over the range in question, one naturally takes the principal value. Now suppose that $\nu = \nu_0$, where ν_0 is in neither of the intervals $(-\infty, -1)$, $(1, \infty)$. Then Eq. (3) implies that ν_0 is the solution of the equation

$$1 = \frac{1-\alpha}{2\nu_0} \cdot \log \frac{1+\nu_0}{1-\nu_0}; \qquad (7)$$

if ν is in one of the above intervals, then Eq. (3) implies that

$$c(\nu) = 1 - \frac{1-\alpha}{2\nu} \log \frac{\nu+1}{\nu-1}. \qquad (8)$$

It is well known that there are precisely two roots of Eq. (7), differing only in sign; thus we have derived two discrete eigensolutions to Eq. (1), plus a continuum of eigensolutions for $\nu \epsilon\, [(-\infty, -1), (1, \infty)]$.

Davison's completeness proof follows.

"The fact that in the 'plane case' the formulae (2) to (8) give a complete set of eigensolutions is known, but its proof does not seem to have been recorded in any report, and therefore, it would appear preferable to give the proof of this fact. This is the purpose of this present appendix.

"The statement that the family of eigensolutions given by (2) to (8) is a complete set of eigensolutions for the plane case, means that any solution valid in an arbitrary infinite slab $a_1 < z < a_2$ with arbitrary boundary conditions at its surfaces $z = a_1$ and $z = a_2$ can be expressed in terms of the eigensolutions given by (2) to (8). Now a solution valid in an infinite slab $a_1 < z < a_2$ can always be thought of as the angular distribution arising from two anisotropic plane sources situated at $z = a_1$ and $z = a_2$; and without any loss of generality we can put $a_1 = 0$ and replace a_2 by a, thus considering the infinite slab $0 < z < a$. Next the angular distribution in this slab can be considered as the superposition of the angular distribution in the half space $z > 0$, due to some appropriate plane source at $z = 0$, and of the angular distribution in the half space $z < a$, due to some appropriate plane source at $z = a$. This for the case of $\alpha \geqslant 0$ (no multiplication) can be seen, say, from the following considerations. Let $q_{o1}(\mu)$ be the angular distribution of the particles emitted by the (original) plane source at $z = 0$, and let $q_{o2}(\mu)$ be the angular distribution of the particles emitted by the (original) plane sources at $z = a$. Let also $\Psi^{(0)}(r,\mu)$ be that angular distribution in the half space $z > 0$ due to the plane source $q_{o1}(\mu)$ which remains bounded as z tends to infinity. This angular distribution will differ from the correct angular distribution in the slab, $\Psi(r,\mu)$, say because it does not contain the contribution from the plane source $q_{o2}(\mu)$ and because it contains also the particles coming from the region $z > a$, which in the correct angular distribution $\Psi(r,\mu)$ would be absent. These particles can be replaced by particles coming from some anisotropic plane source $q_{11}(\mu)$ say, situated at $z = a$. Put now $q_1(\mu) = q_{o2}(\mu) - q_{11}(\mu)$ and let $\Psi^{(1)}(r,\mu)$ be that angular distribution in the half space $z < a$ due to the anisotropic plane source $q_1(\mu)$ situated at $z = a$, which remains bounded as z tends to minus infinity. This angular distribution $\Psi^{(1)}(r,\mu)$ will differ from the correction $\{\Psi(r,\mu) - \Psi^{(0)}(r,\mu)\}$ which should

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[1] B. DAVISON, "Angular Distribution Due to an Isotropic Point Source and Spherically Symmetrical Eigensolutions of the Transport Equation," Canadian Report MT-112, National Research Council of Canada, Division of Atomic Energy (1945).

[2] Davison's report was discovered through a reference by ERWIN H. BAREISS in a report "Decomposition of the Stationary Isotropic Transport Operator in Three Independent Space Variables," ANL-6914, Argonne National Laboratory (1964).

[3] K. M. CASE, "Elementary Solutions of the Transport Equation and their Applications," *Ann. Phys.*, 9,1 (1960).

[4] M. J. LIGHTHILL, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge, London (1958).

be applied to $\Psi^{(0)}(r,\mu)$ in order to obtain the correct angular distribution in the slab, in that respect that $\Psi^{(1)}(r,\mu)$ contains also the particles coming from the region $z < 0$, while the true correction to be applied to $\Psi^{(0)}(r,\mu)$ should not contain those particles. This however can again be corrected by introducing an additional plane source $q_2(\mu)$ at $z = 0$ and constructing the angular distribution due to this plane source in the half space $z > 0$. We can proceed in this manner indefinitely. For $\alpha \geqslant 0$ (no multiplication) this process is obviously convergent, since choosing for $\Psi^{(n)}(r,\mu)$ that angular distribution which remains bounded for z tending to plus (or minus) infinity, we safeguard that this angular distribution does not contain any particles coming directly or indirectly from infinity, but contain only the particles coming directly or indirectly from the source, $q_n(\mu)$. Of these particles only a certain fraction can penetrate beyond the "natural extent" of the slab (i.e. into the region $z > a$ for an even n or the region $z < 0$ for an odd n) as for $\alpha > 0$ some fraction of the particles will be lost through absorption, and whether $\alpha > 0$ or $\alpha = 0$ some fraction will be lost by escaping through the open surface ($z = 0$ for an even n and $z = a$ for an odd n). Thus the number of particles which should be neutralized by the source $q_{n+1}(\mu)$, and consequently the total strength of the source $q_{n+1}(\mu)$ cannot exceed some fixed fraction of the total strength of the source $q_n(\mu)$ for $n \geqslant 1$, so that the series $\sum_{n=1}^{\infty} \Psi^{(n)}(r,\mu)$ should converge at least as a geometrical progression. Thereby the series

$$\sum_{n=0}^{\infty} \Psi^{(n)}(r,\mu) = \Psi(r,\mu) \tag{B.1}$$

will give the correct angular distribution in the slab, while the series

$$\sum_{n=0}^{\infty} \Psi^{(2n)}(r,\mu) = \Psi_{+}(r,\mu) \tag{B.2}$$

will give an angular distribution in the half space $z > 0$ and the series

$$\sum_{n=0}^{\infty} \Psi^{(2n+1)}(r,\mu) = \Psi_{-}(r,\mu) \tag{B.2^1}$$

say, will give an angular distribution in the half space $z < a$. And the relationship

$$\Psi(r,\mu) = \Psi_{+}(r,\mu) + \Psi_{-}(r,\mu) \tag{B.3}$$

confirms our statement that any angular distribution in an infinite slab can be thought of as a superposition of angular distribution in two half spaces. Therefore to establish that the family of eigensolutions of the transport equation for the 'plane case' given by (2) to (8) is complete it is sufficient to establish that the bounded solution in the half-space $z > 0$ due to an arbitrary plane source at $z = 0$ can be built out of the eigensolutions (2) to (8). This can be seen at once as follows. For the bounded solution in the half space $z > 0$ (meaning the solution in which the density is bounded for $z \to +\infty$, but not necessarily bounded near $z = 0$) we can easily derive from the transport equation the inhomogeneous integral equation for the density. Solving this equation by Wiener-Hopf method, and changing the paths of integration we can always represent the resulting expression for the density in the form

$$\Psi_0(Z) = A\, e^{-\nu_0 z} + \int_1^{\infty} F(\nu)\, e^{-\nu z}\, d\nu \tag{B.4}$$

so that the density in our problem is the superposition of densities corresponding to the eigensolutions (2) to (8). On the other hand while we are dealing with a stationary (time independent) solution in a half space, possessing a bounded density as z tends to infinity, the complete angular distribution is uniquely defined as soon as we are given the law of density variation and to obtain this unique angular distribution we can, for instance, superimpose the angular distribution corresponding to the densities appearing in (B.4) and obtain

$$\Psi(z,\mu) = A\, f_{\nu_0}(\mu) e^{-\nu_0 z} + \int_1^{\infty} F(\nu) f_{\nu}(\mu) e^{-\nu z}\, d\nu \tag{B.5}$$

in which $f_{\nu_0}(\mu)$ and $f_{\nu}(\mu)$ are given by (6) to (8). Returning to the case of an infinite slab we shall obviously have, instead of (B.4) the expression

$$\Psi_0(z) = A^{+} e^{-\nu_0 z} + A^{-} e^{\nu_0 z} + \int_1^{\infty} F^{+}(\nu) e^{-\nu z}\, d\nu + \int_1^{\infty} F^{-}(\nu) e^{\nu z}\, d\nu \tag{B.6}$$

with the corresponding modification of the formula (B.5).

"In the entire above argument we were assuming that $\alpha \geqslant 0$. If α is negative the integral equation for the density possesses periodic eigensolutions and the argument which has lead us to the splitting of (B.3) will break down in two respects. Firstly the bounded solution in a half space satisfying the given boundary conditions on its limiting plane is no longer unique, and, if we pick one of these bounded solutions somehow, the process described in connection with (B.3) need no longer be convergent. Also, for $\alpha < 0$, there exists for any fixed value of α, an infinite set of the value of a (critical values of thickness) for which the solution for the slab with the given plane sources at its boundaries may be either non-unique, or non-existent altogether. However, assuming beforehand that we are dealing with a case in which the solution exists and selecting at each step the appropriate one among the bounded solutions in the half space involved, we could again prove the possibility of the representation (B.6), and consequently the completeness of the family of eigensolutions (2) to (8) for the plane case. We shall however refrain from going into detail in this argument, as it is fairly obvious from the general considerations that once the family (2) to (8) is complete for $\alpha \geqslant 0$, it should be also complete for $\alpha < 0$."

Although one could question whether the extension of the results to the case $\alpha < 0$ is really "fairly obvious," the remainder of Davison's proof is surprisingly clear, considering the subtlety of the theorem. It is interesting to note that implicit in Davison's proof is an approach to finite-slab problems involving a combination of the method of images with the Wiener-Hopf technique. Although this would be an iterative technique, so too is Case's method which, for finite systems, involves the iterative solution of a Fredholm equation. Perhaps the two methods are equivalent!

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