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Letter to the Editor

The Albedo Problem and Chadrasekhar's *H*-Function II*

This extension of our recent letter[1] (which will here be referred to as I) gives more accurate analytic approximations to Chandrasekhar's H-function, and to the directional and net reflection functions (albedos) for a semi-infinite medium with isotropic scattering. Following a suggestion of P. Rafalski[2], the relationship of our work[1,3] to that of Romanov[4] is also discussed.

In I we approximated the H-function[5]

$$H(\mu) = 1 + \frac{c}{2}\,\mu H(\mu)\int_0^1 \frac{H(\mu')d\mu'}{\mu + \mu'} \tag{1}$$

by its first iterates $H_{ji}^{(1)}$ resulting from particular choices of initial (zero-) approximations $H_{ji}^{(0)}$ satisfying the conditions

$$H(0) = 1, \qquad [H(\mu)]_{c=0} = 1 \tag{2a,b}$$

$$\frac{c}{2}\int_0^1 H(\mu) = 1 - (1-c)^{1/2}. \tag{3}$$

The subscript i takes values[1] 1, 2, and 3, each of which refers to a different iterative formula. In particular, the value $i = 2$ refers to the iterative formula most suited for our present purpose

$$1/H_2^{(n+1)}(\mu) = 1 - \frac{c}{2}\,\mu\int_0^1 \frac{H_2^{(n)}(\mu')d\mu'}{\mu + \mu'}\,. \tag{4}$$

We here observe that the H-function satisfies the additional condition[6]

$$\frac{c}{2}\int_0^1 \frac{H(\mu)}{1 - K\mu}\,d\mu = 1\,, \tag{5}$$

where K is the positive root of the equation

$$\frac{c}{2K}\ln\frac{1+K}{1-K} = 1\,. \tag{6}$$

*Work done under the auspices of the USAEC.

[1] I. K. ABU-SHUMAYS, "The Albedo Problem and Chandrasekhar's H-function," *Nucl. Sci. Eng.*, **26**, 430 (1966).

[2] P. RAFALSKI, Private Communication (1966).

[3] I. K. ABU-SHUMAYS, "Generating Functions and Transport Theory," Thesis, Harvard University, Cambridge, Massachusetts (Feb. 1966).

[4] YU. A. ROMANOV, "Exact Solutions of Single Velocity Kinetic Equation and their Application in Calculating Diffusion Problems (Improved Diffusion Method)," FTD-TT-61-124, p. 2 (1962). Translated from "Issledovasiya Kriticheskikh Parametrov Reaktomykh Sistemy." Also see *Nucl. Sci. Abstracts*, 16, 11, Abs. 14243, p. 1831 (1962).

[5] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford, New York (1950).

[6] SOBOLEV, *A Treatise on Radiative Transfer*, Translated by S. C. Gaposhkin, p. 81, Eq. (114), Van Nostrand (1963).

Conditions of Eqs. (2), (3), and this additional condition of Eq. (5) admit the selection of initial approximations $H_{ji}^{(0)}$ more appropriately than in I, such as

$$H_{3i}^{(0)} = 1 + A\mu + B\mu^2 \tag{7}$$

$$H_{4i}^{(0)} = \frac{a + b\mu}{1 + K\mu} \tag{8}$$

$$H_{5i}^{(0)} = \frac{1 + a'\mu + b'\mu^2}{1 + K\mu}\,. \tag{9}$$

The constants on the right-hand sides of Eqs. (7), (8), and (9), determined from Eqs. (3) and (5) are

$$A = \frac{4}{c}[1 - (1-c)^{1/2}] - 2 - \frac{2}{3}B \equiv \alpha - \frac{2}{3}B \tag{7a}$$

$$B = \frac{K\ln(1+K) + [K + \ln(1-K)]\,\alpha}{\frac{K}{b} + \left(\frac{2}{3} - \frac{1}{K}\right)\ln(1-K) - 1} \tag{7b}$$

$$a = \frac{(1-c)^{1/2}\,b}{K}, \quad b^{-1} = \frac{(1-c)^{1/2}}{2K} - \frac{c}{4K^2}\ln(1-K^2) \tag{8a,b}$$

$$a' = \frac{N}{D}, \quad b' = \left[K^2 + \frac{c}{a}a'\ln(1-K^2)\right]\Big/(1-c) \tag{9a,b}$$

$$N = \frac{2K^2}{c}[1 - (1-c)^{1/2}] - K\ln(1+K) - \frac{K^2}{1-c}\left[\frac{K}{2} - 1 + K^{-1}\ln(1+K)\right]$$

$$D = K - \ln(1+K) + \frac{c\ln(1-K^2)}{2(1-c)}\left[\frac{K}{2} - 1 + K^{-1}\ln(1+K)\right].$$

With the choices of Eqs. (7), (8), and (9), the first iterates of Eq. (4) are, respectively,

$$H_{32}^{(1)} = \left\{1 - \frac{c}{2}\left[(1-A\mu+B\mu^2)\mu\ln\frac{1+\mu}{\mu} + A\mu + \left(\frac{\mu}{2} - \mu^2\right)B\right]\right\}^{-1} \tag{10}$$

$$H_{42}^{(1)} = \left(1 - \frac{cb}{2K(1-K\mu)}\left\{[(1-c)^{1/2} - K\mu]\,\mu\ln\frac{1+\mu}{\mu} + [1 - (1-c)^{1/2}]\mu\ln(1+K)\right\}\right)^{-1} \tag{11}$$

$$H_{52}^{(1)} = \left\{1 - \frac{c}{2(1-K\mu)}\left[(1-a'\mu+b'\mu^2)\,\mu\ln\frac{1+\mu}{\mu} - \left(1 - \frac{a'}{K} + \frac{b'}{K^2}\right)\mu\ln(1+K) + B\mu\left(\frac{1}{K} - \mu\right)\right]\right\}^{-1}. \tag{12}$$

The approximations Eqs. (10), (11), and (12) are superior to those given in I. The term $H_{32}^{(1)}$ has an error less than 0.09% for $c \approx 1$, less than 0.05% for $c = 0.6$, and less than 0.02% for $c \le 0.3$; $H_{42}^{(1)}$ and $H_{52}^{(1)}$ have comparable errors less than 0.09% to $c \approx 1$, less than 0.04% for $c = 0.6$, and less than 0.01% for $c \leq 0.3$. However, $H_{42}^{(1)}$ overestimates H while $H_{52}^{(1)}$ underestimates H. The average of $H_{42}^{(1)}$ and $H_{52}^{(1)}$ is a remarkably accurate approximation to the H-function. It is compared in Table I with values of H-function given by Chandrasekhar[5] (for $c = 0.1$) and Carlstedt and Mullikin[7] ($c \geq 0.3$), and the agreement is better than 0.005% for $c \approx 1$ and better than 0.001% for $c \leq 0.5$. The H-functions given by Carlstedt and Mullikin[7] are presumably accurate to within about 0.001%.

P. Rafalski[2] has pointed out that the method of approximation used here and in I is closely allied to that of Yu. A. Romanov[4] and suggested comparing the results of Romanov with those in I. Romanov[4] uses the equation for the angular distribution function $\phi(\mu)$ on the boundary of a semi-infinite isotropic medium (Milne Problem)

$$\int_0^1 \frac{\phi(\mu')\mu' d\mu'}{\mu + \mu'} = \frac{c}{2\phi(\mu)(1-K^2\mu^2)}, \tag{13}$$

where K is the root of Eq. (6). He approximates $\phi(\mu)$ by its first iterate obtained using an initial approximation

$$\phi_0(\mu) = \frac{a + b\mu}{1 - K^2\mu^2} . \tag{14}$$

TABLE I

The H-function and its Analytic Approximation

μ \ c	0	0.1	0.3	0.5	0.7	0.9	0.99	1.0
				$H \approx \frac{1}{2}[H_{42}^{(1)} + H_{52}^{(1)}]$				
0	1	1	1	1	1	1	1	1
0.2	1	1.01864	1.06116	1.11348	1.18255	1.29148	1.39982	1.45041
0.4	1	1.02630	1.08811	1.16798	1.28062	1.47848	1.70748	1.82925
0.6	1	1.03106	1.10538	1.20434	1.35006	1.62584	1.98328	2.19406
0.8	1	1.03436	1.11762	1.23088	1.40287	1.74734	2.23690	2.55260
1.0	1	1.03682	1.12684	1.25125	1.44472	1.85003	2.47268	2.90768
				H-function				
0	1	1	1	1	1	1	1	1
0.2	1	1.01864	1.06115	1.11346	1.18252	1.29143	1.39977	1.45035
0.4	1	1.02630	1.08811	1.16797	1.28062	1.47850	1.70750	1.82928
0.6	1	1.03106	1.10537	1.20435	1.35008	1.62588	1.98336	2.19413
0.8	1	1.03436	1.11763	1.23089	1.40290	1.74740	2.23700	2.55270
1.0	1	1.03681	1.12684	1.25126	1.44475	1.85010	2.47279	2.90781

We observe that the H-function and the angular distribution function ϕ are related by

$$H(\mu) = \frac{2}{c}(1 - K\mu)\phi(\mu) \tag{15}$$

as can easily be demonstrated from Eqs. (1) and (13). We further observe that with Eq. (15), the expressions[1,5,8] for the directional and net albedos in terms of H and in terms[4] of ϕ are equivalent. The first iterate ϕ_1 from Eqs. (13) and (14) is, in fact, equivalent to Eqs. (8) and (11).

The above approximations for the H-function when applied to the expressions for the directional and net reflection functions in terms of the H-function[1,5,8] yield the desired approximations. In particular, for particles incident in the direction μ_0, the net albedo is

$$R(\mu_0) = 1 - (1-c)^{1/2} H(\mu_0). \tag{16}$$

Table II compares $1-R(1)$ and its approximate value achieved by replacing H by $\frac{1}{2}[H_{42}^{(1)} + H_{52}^{(1)}]$.

TABLE II

$1 - R(\mu_0 = 1) = (1-c)^{1/2} H(1)$

c	0	0.1	0.3	0.5	0.7	0.9	0.99	1
$\frac{1}{2}[H_{42}^{(1)} + H_{52}^{(1)}]$	1	0.98361	0.94278	0.88477	0.79131	0.58503	0.24727	0
H-function	1	0.98360	0.94278	0.88477	0.79132	0.58505	0.24728	0

A more complete comparison[4] of the various approximations to the H-function and directional and net reflection functions will be presented as an ANL report. Further applications of the above approximations are being investigated.

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[7] J. L. CARLSTEDT and T. W. MULLIKIN, "Chandrasekhar's X and Y Functions," *Astrophys. J. Suppl.*, 12, 113 (1966).

[8] D. S. SELENGUT, "Distribution of Neutrons Reflected from a Semi-infinite Slab," *Reactor Technology*, KAPL-2000-20, Report No. 23, p. III. 44, Knolls Atomic Power Laboratory (1963).

Corrigendum

BAL RAJ SEHGAL, "Monte Carlo Calculations of Resonance Integral of ^{232}Th," *Nucl. Sci. Eng.*, **27**, 95 (1967).

The first equation on p. 102 should read:

$$(S/M)_{Th} = 1.138\,(S/M)_{ThO_2} + 0.066.$$