at the diffusion plant as it brings to a close its history of 40 years of productive operation.

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Comments on the Richardson Extrapolation

In a recent letter to the editor,[1] Makai pointed out the efficiency of the Richardson extrapolation method[2] in obtaining finite difference (FD) solutions of higher accuracy from two FD solutions of lower accuracy. We confirm the efficiency and accuracy of this method, as shown by a number of calculations we have made[3] for various fast reactor configurations using two- and three-dimensional multigroup FD neutron diffusion computer codes in *r-z*, *x-y*, and *x-y-z* geometries. Good results have been reported by others, as well (for example, see Refs. 4 and 5).

The purpose of this letter to the editor is to bring out some other points concerning the application of this method in reactor calculations:

1. Not only is the computer time reduced by the use of this extrapolation procedure but, perhaps more importantly, the memory storage requirements are greatly reduced. It should be recalled that for a three-dimensional FD problem the computer memory requirements increase as the cube of the number of meshes. We made particular use of the Richardson extrapolation method in problems where the fine mesh case just did not fit into the computer.

2. The method could fail when the higher order terms of the discretization error series are not negligibly small. Hence, the applicability of the procedure to each class of problems must be separately established before routine use. This can most easily be checked by making a series of calculations with gradually finer meshes. A straightforward Richardson extrapolation of two coarse-mesh solutions should not be made if the successive approximations do not approach the true value monotonically, since in this case the higher terms of the error series cannot be neglected.

3. By having solutions with three different mesh sizes, it is possible to have an estimate of the order j of the first term of the discretization error series. For example, with three solutions u_1, u_2, and u_3 corresponding to mesh sizes h, $2h$, and $4h$, we have

$$j \approx \frac{\ln\left(\dfrac{u_2 - u_3}{u_1 - u_2}\right)}{\ln 2} .$$

4. For the conventional FD neutron diffusion equation, the truncation error is of $O(h^2)$ if there is a single homogeneous region and uniform mesh width. However, at boundaries between different regions in a reactor, the truncation error becomes of $O(h)$. Hence, if there are a large number of different regions or nonuniform mesh widths in the considered problem, an h^2 extrapolation of two coarse-mesh solutions may not be valid.

5. There is a slight discrepancy in Ref. 1. In Eqs. (3) and (6), the first term of the discretization error series has been indicated as $j = 1$. However, the weighting factors $\nu_1 = -\frac{1}{3}$ and $\nu_2 = \frac{4}{3}$ that have been used in the case study correspond to a single term error series starting with $j = 2$.

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Reply to "Comments on Richardson Extrapolation"

The advantages of applying Richardson extrapolation can be summarized as follows:

1. A solution of given accuracy is cheaper due to the larger mesh size, which leads to a faster algorithm, reduced memory requirement, and larger numerical stability.
2. The method is flexible; applications to *r-z*, *x-y*, *x-y-z* (see Refs. 1 and 2) and hexagonal geometry[3,4] have been reported.
3. The method is applicable to a large number of problems, including finite difference[2-4] (FD), finite element,[5] coarse-mesh, and diverse transport theory methods. Application to the S_N method has been reported in Ref. 6.

Let us return to the FD method and pay attention to the problem of extrapolating k_{eff}. By definition,

$$k_{eff} = \frac{\text{production}}{\text{absorption} + \text{leakage}} , \qquad (1)$$

where both production and absorption are integrals over the core, while leakage is an integral over the surface of the core. Some difficulty is caused by the FD method's having different truncation errors at material boundaries and in homogeneous regions, excluding the legitimacy of extrapolating the nominator and denominator of Eq. (1) in the same way.[2] That approximation, though not correct, is often useful. It is even more convenient to assume k_{eff} to behave as any reaction rate[3] in a homogeneous region and to express the truncation error as

$$k_{eff} = K_{eff} + a \cdot h^2 , \qquad (2)$$

TABLE I

WWER-1000 Benchmark ($h = 14$ cm) Results by HEXAGA-III

i	h_i[a]	Points/ Hexagon	k_{eff}	Time (s, CYBER 73)
1	1	3	1.11715	6
2	2	12	1.11375	14
3	3	27	1.11279	34
4	4	48	1.11240	60
5	6	108	1.11211	237

[a]The value h_i is mesh size in relative units.

where h is the mesh size. We would recommend here a useful method proposed by Koch.[7] The problem considered below is a WWER-1000 benchmark[8] and the results were obtained by the HEXAGA program.[9]

Table I shows the eigenvalues k_{eff} obtained at five mesh sizes. The extrapolation procedure is given in Table II. The proposed value is

$$K_{eff} = 1.11187 \pm 0.00002 \ ,$$

which is just k_{15} and the error is $(k_{15} - k_{14})$.

All in all, the reader is encouraged to use the Richardson extrapolation.

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TABLE II

Extrapolation of k_{eff} by Koch's Method

h_i	k_i			
1	1.11715			
2	1.11375[a]	$k_{12} = p_1k_2 + p_2k_1 = 1.11262$		
3	1.11279	$k_{23} = p_3k_3 + p_4k_2 = 1.11202$	$k_{13} = v_1k_{23} + v_2k_{12} = 1.11195$	
4	1.11240	$k_{34} = p_5k_4 + p_6k_3 = 1.11190$	$k_{24} = v_3k_{34} + v_4k_{23} = 1.11186$	$k_{14} = u_1k_{24} + u_2k_{13} = 1.11185$
5	1.11211	$k_{45} = p_7k_5 + p_8k_4 = 1.11188$	$k_{35} = v_5k_{45} + v_6k_{34} = 1.11187$	$k_{25} = u_3k_{35} + u_4k_{24} = 1.11187$
				$k_{15} = w_1k_{25} + w_2k_{14} = 1.11187$

[a] $p_1 = \frac{4}{3}$; $p_2 = -\frac{1}{3}$; $p_3 = \frac{9}{5}$; $p_4 = -\frac{4}{5}$; $p_5 = \frac{16}{7}$; $p_6 = -\frac{9}{7}$; $p_7 = \frac{9}{5}$; $p_8 = -\frac{4}{5}$

$v_1 = \frac{9}{8}$; $v_2 = -\frac{1}{8}$; $v_3 = \frac{4}{3}$; $v_4 = -\frac{1}{3}$; $v_5 = \frac{4}{3}$; $v_6 = -\frac{1}{3}$

$u_1 = \frac{16}{15}$; $u_2 = -\frac{1}{15}$; $u_3 = \frac{9}{8}$; $u_4 = -\frac{1}{8}$

$w_1 = \frac{36}{35}$; $w_2 = -\frac{1}{35}$

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