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Letters to the Editor

The Albedo Problem and Chandrasekhar's H-Function*

In recent notes, Rafalski[1] and Pomraning[2] have developed analytic approximations for the albedo problem (reflection functions) for a semi-infinite medium with isotropic scattering.

Alternative, more accurate analytic approximations to the albedo problem are suggested below. The directional and net reflection functions (albedos) are related to Chandrasekhar's H-function as in his Eq. (109), p. 124 (Chandrasekhar[3], Abu-Shumays[4]). In particular, for incident particles with $\mu = \mu_0$ (with respect to the inward normal to the surface), the net albedo is

$$R(\mu_0) = 1 - \sqrt{1-c}\, H(\mu_0), \tag{1}$$

where c is the number of secondaries per collision, and for isotropic incidence, the net albedo is

$$\overline{A}_{is} = 1 - 2\sqrt{1-c}\int_0^1 H(\mu)\mu d\mu \equiv 1 - 2\alpha_1\sqrt{1-c}. \tag{2}$$

The desired expressions are obtained by approximating Chandrasekhar's[3] H-function

$$H(\mu) = 1 + \frac{c}{2}\mu H(\mu)\int_0^1 \frac{H(\mu')d\mu'}{\mu + \mu'}, \quad 0 \leqslant c \leqslant 1 \tag{3}$$

by its first iterates $H^{(1)}$,s (see below) resulting from particular choices of initial approximations $H^{(0)}$,s [Eqs. (8), (11), and (12)] and suitable arrangements of the H-equation[5]. The iterative formulae are[4,5]

$$H_1^{(n+1)}(\mu) = 1 + \frac{c}{2}\mu H_1^{(n)}(\mu)\int_0^1 \frac{H_1^{(n)}(\mu')d\mu'}{\mu + \mu'} \tag{4}$$

$$1/H_2^{(n+1)}(\mu) = 1 - \frac{c}{2}\mu\int_0^1 \frac{H_2^{(n)}(\mu')d\mu'}{\mu + \mu'} \tag{5}$$

$$1/H_3^{(n+1)}(\mu) = \sqrt{1-c} + \frac{c}{2}\int_0^1 \frac{H_3^{(n)}(\mu')\mu' d\mu'}{\mu + \mu'}. \tag{6}$$

The H-function satisfies[3] $H(0) = 1$ and

$$\frac{c}{2}\int_0^1 H(\mu)d\mu = 1 - \sqrt{1-c}. \tag{7}$$

Equation (4) has the following advantage: If the initial approximation $H^{(0)}$ satisfies Eq. (7), then all iterates satisfy Eq. (7).

A plausible choice to start the iteration is the average value of H which, from Eq. (7), is given by

$$H_{1i}^{(0)} = \overline{H} = \frac{2}{c}(1 - \sqrt{1-c}), \tag{8}$$

where i = 1, 2, 3 refers, respectively, to the iterative formulae of Eqs. (4), (5), and (6). This choice leads to

$$H_{11}^{(1)} = 1 + \left[\frac{4}{c}(1 - \sqrt{1-c}) - 2\right]\mu \ln\frac{1+\mu}{\mu} \tag{9}$$

$$H_{12}^{(1)} = H_{13}^{(1)} = \left[1 - (1 - \sqrt{1-c})\mu \ln\frac{1+\mu}{\mu}\right]^{-1}. \tag{10}$$

The approximations $H \approx H_{1i}^{(1)}$ are accurate for small c (c close to zero). $H \approx H_{12}^{(1)}$ is the better of the above two approximations and is remarkably accurate for some values of c and μ. It has an error less than 0.04% for $c \leqslant 0.2$, less than 0.2% for $c \leqslant 0.4$, less than 0.6% for $c \leqslant 0.6$, and less than 1.7% for $c \leqslant 0.8$. However, $H \approx H_{12}^{(1)}$ is inadequate for $c = 1$ (error 7%).

Another plausible choice to start the iteration is a linear function of μ

$$H_{2i}^{(0)} = 1 + \alpha\mu, \qquad i = 1,2,3, \tag{11}$$

with α selected so as to preserve the average value of $H(\mu)$ in Eq. (7), i.e.,

$$\alpha = \frac{4}{c}(1 - \sqrt{1-c}) - 2. \tag{12}$$

This choice [Eqs. (11) and (12)] leads to

$$H_{21}^{(1)} = 1 + (1+\alpha\mu)\left[\frac{\alpha c}{2}\mu + (1-\alpha\mu)\frac{c}{2}\mu \ln\frac{1+\mu}{\mu}\right] \tag{13}$$

$$H_{22}^{(1)} = H_{23}^{(1)} = \left[1 - \frac{\alpha c}{2}\mu - (1-\alpha\mu)\frac{c}{2}\mu \ln\frac{1+\mu}{\mu}\right]^{-1}. \tag{14}$$

Equation 13 is already superior to the approximations $H \approx H_{1i}^{(1)}$, having a maximum error of 2% for $c = 1$; $H_{21}^{(1)}$ underestimates H for μ close to zero and overestimates H for μ close to 1. ($H_{11}^{(1)}$ has the opposite behavior.) Consequently, it supplies excellent approximations to H at μ close to 0.5 or 0.6.

Equation (14) is the best when the full range of $0 \leqslant c \leqslant 1$ and $0 \leqslant \mu \leqslant 1$ is taken into consideration. However, it is slightly inferior to $H_{11}^{(1)}$, $H_{21}^{(1)}$ at μ close to 0.5 or 0.6 and a few values of c justifiable as in the case of $H \approx H_{21}^{(1)}$ of the previous paragraph. Computed values of $H_{22}^{(1)}$ are given in Table I together with values of H-function from Table XI of Chandrasekhar[3].

Further, of all $H_{ij}^{(1)}$, $H_{11}^{(1)}$ and $H_{21}^{(1)}$ supply the simplest analytic approximations whenever integrated forms of the H-function such as the moments of the H-function are desired. The zero'th moment [Eq. (7)] is given exactly by

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[1]P. RAFALSKI, *Nucl. Sci. Eng.*, 19, 3, 378 (1964); I have shown (private work) that, according to Rafalski's approximation, his result for the albedo is not restricted to normal incidence.

[2]G. C. POMRANING, *Nucl. Sci. Eng.*, 21, 1, 62 (1965).

[3]S. CHANDRASEKHAR, *Radiative Transfer*, Oxford (1950).

[4]I. ABU-SHUMAYS, "Generating Functions and Transport Theory," Thesis, Harvard University, Cambridge, Massachusetts (1966).

[5]B. NOBLE, "The Numerical Solution of Nonlinear Integral Equations and Related Topics," *Nonlinear Integral Equations*, P. M. Anselone, Ed., University of Wisconsin Press, Madison, Wisconsin (1964).